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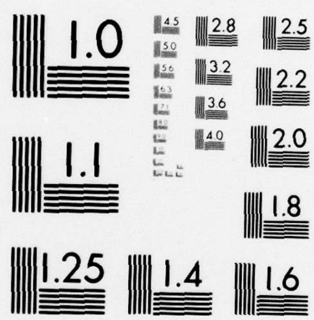

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The Constraint Method for Solid Finite Elements

Annual Technical Report, October 1, 1978 - September 30, 1979

by

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The p-version of the finite element method is a new approach to finite element analysis which has been demonstrated to lead to significant computational savings, often by orders of magnitude (This approach was formerly called the constraint method; the new term p-version is more descriptive). Conventional approaches (called the h-version) generally employ low order polynomicals as basis functions. Accuracy is achieved by suitably refining the approximating mesh. The p-version uses polynomials of arbitrary order p greater than or equal to 2 for problems in plan elasticity where CO continuity

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is required and polynomials of order p greater than or equal to 5 for problems in plate bending where C1 continuity is required.

Hierarchic elements which implement the p-version efficiently are used together with precomputed arrays of elemental stiffness matrices. C0 solid elements of various shapes have been formulated.

A major result that has recently been obtained on the convergence of the p-version of the finite element method is: in polynomial regions, the p-version converges approximately twice as fast as the h-version.

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Discussion

The p-version of the finite element method is a new approach to finite element analysis which has been demonstrated to lead to significant computational savings, often by orders of magnitude (This approach was formerly called the constraint method; the new term p-version is more descriptive). Conventional approaches (called the h-version) generally employ low order polynomials as basis functions. Accuracy is achieved by suitably refining the approximating mesh. The p-version uses polynomials of arbitrary order $p \geqslant 2$ for problems in plane elasticity where C0 continuity is required and polynomials of order $p \geqslant 5$ for problems in plate bending where C1 continuity is required.

Hierarchic elements which implement the p-version efficiently are used together with precomputed arrays of elemental stiffness matrices.

Major accomplishments during this past year are summarized in the following three documents which are enclosed:

1. "Comparative Rates of h- and p- convergence in the Finite element Analysis of a Model Bar Problem" by I. Norman Katz (abstract), presented at SIAM 1978 Fall Meeting, October 30, 31, November 1, 1978 in Knoxville, Tennessee.
2. "The p-Version of the Finite Element Method" by I. Babuska, B. S. Szabo, and I. Norman Katz (Report, submitted for publication), Report WU/CCM-79/1, May 1979
3. "Hierarchic Families for the p-Version of the Finite Element Method", by I. Babuska, I. Norman Katz and B. A. Szabo, Proceedings of the Third IMACS International Symposium on Computer Methods for Partial Differential Equations, Lehigh University, Bethlehem, PA, June 20 - 22, 1979

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I. Norman Katz, Washington University, St. Louis, Missouri 63130

Comparative Rates of h- and p-convergence in the Finite Element Analysis of a Model Bar Problem

The conventional approach to finite element stress analysis of a body defined by a polygonal domain Ω (in two dimensions) is to triangulate Ω and to seek accuracy by letting h, the maximum diameter of all elements in the triangulation, tend to zero. This approach, called h-convergence, has been the subject of intensive investigation. Another approach which is being developed at Washington University is to fix the triangulation of Ω and to let p, the degree of the complete, conforming, approximating polynomial over each triangle, tend to infinity. Extensive numerical tests have shown that the second approach, called p-convergence, is considerably more accurate than the first, even in problems whose solutions have singularities such as cracks or corners.
In order to illustrate the comparative rates of convergence, a model (one-dimensional) bar problem is studied. Asymptotic analysis leads to expressions for the rates of convergence in the two approaches, when the solution possesses a singularity which is known a priori. It is demonstrated that the order of p-convergence is twice that of h-convergence, provided that the singularity is located at some node of a finite element.

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THE P-VERSION OF THE FINITE ELEMENT METHOD

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ABSTRACT

In the p-version of the finite element method the triangulation is fixed and the degree p of the piecewise polynomial approximation is progressively increased until some desired level of precision is reached.

In this paper we first establish the basic approximation properties of some spaces of piecewise polynomials defined on a finite element triangulation. These properties lead to an a priori estimate of the asymptotic rate of convergence of the p-version. The estimate shows that the p-version gives results which are not worse than those obtained by the conventional finite element method (called the h-version, in which h represents the maximum diameter of the elements) when quasi-uniform triangulations are employed and the basis for comparison is the *number of degrees of freedom*. Furthermore, in the case of a singularity problem we show (under conditions which are usually satisfied in practice) that the rate of convergence of the p-version is twice that of the h-version with quasi-uniform mesh. Inverse approximation theorems which determine the smoothness of a function based on the rate at which it is approximated by piecewise polynomials over a fixed triangulation are proved both for singular and non-singular problems.

We present numerical examples which illustrate the effectiveness of the p-version for a simple one dimensional problem and for two problems in two dimensional elasticity. We also discuss round off error and computational costs associated with the p-version. Finally we describe some important features, such as hierarchic basis *functions*, which have been utilized in COMET-X, an experimental computer implementation of the p-version.

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1. INTRODUCTION

The finite element method, one of the most widely used numerical methods for solving certain types of differential equations, is based on approximating the solution by piecewise smooth functions, specifically polynomials, on convex subdomains such as triangles. In general, the degree of the polynomials is fixed at some arbitrarily chosen low number. No consensus exists at the present time concerning the most suitable (optimal) degree p of the polynomials.

The mathematical justification of the finite element method is based on asymptotic analyses in which p is kept bounded and the diameters of the element subdomains approach zero. However, it has been observed by several investigators that the sizes of elements used in practical computations are often outside of the range of asymptotic behavior.

Because the maximum diameter of finite elements is usually denoted by h, we shall refer to this (conventional) approach as the <u>h-version</u> of the finite element method.

From the theoretical point of view one can justify the finite element method, also in the asymptotic sense, when the subdomains are kept constant and the degree of the approximating polynomials tends to infinity. We shall refer to this method of approximation as the <u>p-version</u> of the finite element method.

The p-version of the finite element method is similar to the Ritz method but there is one very important difference: In the p-version of the finite element method the domain of interest is divided into convex subdomains and the polynomial approximants are piecewise smooth only over individual convex subdomains. In the Ritz method, on the other

hand, the solution over the entire domain is approximated by smooth functions. This difference accounts for the greater versatility and higher rate of convergence of the p-version of the finite element method over both the Ritz method, and the h-version of the finite element method, as demonstrated here.

In this paper we analyze the p-version of the finite element method and its theory, and discuss the implementation characteristics of the method based on the computer program COMET-X, developed during the last few years at Washington University in St. Louis. We also examine the potential for further development of the p-version. We remark that, from the computational point of view, and from the point of view of the architecture of the computer program, there are significant differences between the p-version when p is in the range of 6,7,8 and the h-version when p is in the range 1,2,3.

We present a proof for the rate of convergence in the p-version and show that the polynomials are able to "absorb" singularities, including e.g., corner singularities, when they are located at the vertices of triangles. This does not occur when the corner singularities are not located at vertices.

Comparison of the asymptotic behavior of the h-version, based on uniform or quasi-uniform mesh refinement on one hand, and the p-version on the other, the basis of comparison being the number of degrees of freedom, shows that the rate of convergence of the p-version cannot be slower than the rate of convergence of the h-version and, furthermore, when corner singularities are present at vertices, the rate of convergence of the p-version is exactly twice that of the h-version.

2. BASIC NOTATIONS

Throughout this paper R^2 will be the two dimensional Euclidean space $(x_1,x_2) \equiv x \in R^2$, $\Omega \subset R^2$ will be a bounded domain with piecewise smooth boundary $\partial\Omega$. In particular we will deal with polygonal domains (We exclude - for technical reasons - the slit domain, although the results of this paper can be generalized to this case too with some, but not essential, technical difficulties).

$\mathcal{E}(\bar{\Omega})$ shall be the space of all real C^∞ functions on Ω, with continuous extensions of all derivatives on $\bar{\Omega}$. All functions of $\mathcal{E}(\bar{\Omega})$ with compact support in Ω form a subspace $\mathcal{D}(\Omega) \subset \mathcal{E}(\bar{\Omega})$. As usual, $L_2(\Omega) = H^0(\Omega)$ will be the space of all square-integrable functions on Ω with the inner product.

$$(u,v)_{0,\Omega} = \int_\Omega uvdx\,, \quad dx = dx_1 dx_2$$

and the corresponding norm $||\cdot||_{0,\Omega}$. In addition for any $k \geq 1$, integral, the Sobolev spaces $H^k(\Omega)$ resp $H_0^k(\Omega)$ will be the completions of $\mathcal{E}(\bar{\Omega})$ resp. $\mathcal{D}(\Omega)$ under the norm

$$||u||_{k,\Omega}^2 = \sum_{0 \leq |\alpha| \leq k} ||\mathcal{D}^\alpha u||_{0,\Omega}^2\,,$$

where

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}\,, \qquad \alpha = (\alpha_1,\alpha_2)$$

$\alpha_i \geq 0$ integral $i = 1,2$ and $|\alpha| = \alpha_1 + \alpha_2$. The inner product in $H^k(\Omega)$ will be denoted by $(\cdot,\cdot)_{k,\Omega}$. For $k > 0$ nonintegral the spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are defined by usual interpolation. More precisely for $k = k_0 + \theta$, $0 < \theta < 1$, we define $H^k = [H^{k_0}, H^{k_0+1}]_{\theta,2}$ by application of the usual K-method of interpolation (For more see [7]). [Other notations are $H^k = B^k_{2,2}$ where $B^k_{2,2}$ is the usual Besov space].

For $\rho > 0$ we write

$$Q(\rho) = \{x_1,x_2 \,\big|\, |x_1| < \rho,\ |x_2| < \rho\},$$

$$\hat{Q}(\rho) = \{x_1,x_2 \,|\, 0 < x_1 < \rho,\ 0 < x_2 < \rho\}$$

and by $\mathcal{E}_{PER}(\bar{Q}(\rho)) \subset \mathcal{E}(\bar{Q}(\rho))$ we denote the space of all functions with period 2ρ and by $H^k_{PER}(Q(\rho))$ its closure in $H^k(Q(\rho))$.

We will deal also with Sobolev spaces in one dimension. Analogously as before we will denote by

$$I(\rho) = \{x_1 \,\big|\, |x_1| < \rho\}$$

and $H^k(I(\rho))$, $H_0^k(I(\rho))$, $H^k_{PER}(I(\rho))$ will have the obvious meaning.

Finally we need to introduce the spaces $\mathcal{P}_p(\Omega) \subset \mathcal{E}(\bar{\Omega})$ of all algebraic polynomials of degree not higher than p and $\mathcal{F}_p(Q(\rho))$ (resp. $\mathcal{F}_p(I(\rho))$ the space of all trigonometric polynomials of degree at most p and period 2ρ.

3. THE CONCEPT OF P-CONVERGENCE OF THE FINITE ELEMENT METHOD

3.1 The model problem

We will be interested in the model problem

$$-\Delta u + u = f \quad \text{on} \quad \Omega_0, \qquad f \in H^0(\Omega_0), \tag{3.1}$$

$$\Gamma u = 0 \qquad \text{on } \partial\Omega_0 , \tag{3.2}$$

where Ω_0 is a bounded polygonal domain and $\Gamma u = u$ or $\Gamma u = \frac{\partial u}{\partial n}$. We can easily generalize our results also to other boundary conditions. As usual we will interpret the problem (3.1), (3.2) in a weak sense, namely we seek $u_0 \in H_0^1(\Omega_0)$ resp. $u_0 \in H^1(\Omega_0)$ so that

$$B(u_0,v) = (f,v)_{0,\Omega_0} \tag{3.3}$$

$$\text{for all} \quad v \in H_0^1(\Omega_0) \text{ resp. } v \in H^1(\Omega_0)$$

where we have used the notation

$$B(u_0,v) = (u_0,v)_{1,\Omega_0} . \tag{3.4}$$

u_0 satisfying (3.3) obviously exists and is uniquely determined.

3.2 Description of the p-version of the finite element method

Let S be a (fixed) triangularization of Ω_0, $S = \{T_i\}$, $i = 1,\dots,m$ where T_i are (open) triangles such that $\bigcup_{i=1}^{m} \bar{T}_i = \bar{\Omega}_0$ and $\bar{T}_i$, $\bar{T}_j$ $i \neq j$

have either a common (entire) side or a vertex or $\bar{T}_i \cap \bar{T}_j = \phi$. Denote now by $P_p^{[S]}(\Omega_0) \subset H^1(\Omega_0)$ the subset of all functions $u \in H^1(\Omega_0)$ such that if $u_{(T_i)}$ is the restriction of u on T_i, then we have $u_{(T_i)} \in P_p(T_i)$ i.e. $P_p^{[S]}(\Omega_0)$ consists of all functions which are piecewise polynomials of degree at most p and belong to $H^1(\Omega_0)$. Further let $P_{p,0}^{[S]}(\Omega_0) = P_p^{[S]}(\Omega_0) \cap H_0^1(\Omega_0)$.

The concept of the p-version of the finite element method consists of finding u_p $p = 1,2,\ldots$ $u_p \in P_{p,0}^{[S]}(\Omega_0)$ (resp. $P_p^{[S]}(\Omega_0)$) (for the boundary condition $\Gamma u = 0$ resp. $\Gamma u = \frac{\partial u}{\partial n}$) so that (3.3) holds for any $v \in P_{p,0}^{[S]}(\Omega_0)$ (resp. $P_p^{[S]}(\Omega_0)$).

Study of the p-version of the finite element method was initiated at the School of Engineering and Applied Science of Washington University in St. Louis [25] in 1970. It has been implemented there in various aspects of stress analysis with very good results, particularly in connection with linear elastic fracture mechanics. Development of the p-version is continuing at the Center for Computational Mechanics at Washington University.

3.3 The basic approximation properties of $P_p^{[S]}(\Omega_0)$ and $P_{p,0}^{[S]}(\Omega_0)$.

THEOREM 3.1. Let $u \in H^k(\Omega_0)$. Then there exists a sequence $z_p \in P_p^{[S]}(\Omega_0)$, $p = 1,2\ldots$ such that for any $0 \le \ell \le k$ (ℓ,k not necessarily integral)

$$||u-z_p||_{\ell,\Omega_0} \le Cp^{-(k-\ell)}||u||_{k,\Omega_0} \tag{3.5}$$

where C is independent of u and p (it depends e.g. on ℓ and k etc.).

Proof. The proof is a standard one. First we prove it for ℓ and k integral. We will construct $z_p \in P_p(\Omega_0)$ such that (3.5) is satisfied.

Assume that $\Omega_0 \subset Q(\rho_0)$. Because Ω_0 is a polygon, it is a Lipschitzian domain and therefore there exists an extension U of $u \in H^k(\Omega_0)$ on $Q(2\rho_0)$ such that supp $U \subset Q(3/2\ \rho_0)$ and

$$||U||_{k,Q(2\rho_0)} \leq C||u||_{k,\Omega_0} \tag{3.6}$$

with C independent of u. As usual we have U = Tu where T is a linear mapping of $H^0(\Omega_0)$ into $H^0(Q(2\rho_0))$, (see e.g. [24]) (which also maps $H^0(\Omega_0)$ into $H^0(Q(2\rho_0))$.

Now let Φ be the (one to one) mapping of $Q(\pi/2)$ onto $Q(2\rho_0)$ determined by the transformation of coordinates $\xi = (\xi_1,\xi_2) \in Q(\pi/2)$, $x = (x_1,x_2) \in Q(2\rho_0)$

$$x_i = 2\rho_0 \sin \xi_i \qquad i = 1,2 \tag{3.7}$$

written in the form

$$\Phi(\xi) = x. \tag{3.8}$$

Let

$$V(\xi) = U(\Phi(\xi))$$

and let

$$\tilde{Q} = \Phi^{[-1]}[Q(3/2\ \rho_0)] \subset Q(\pi/2)$$

($\Phi^{[-1]}$ is the inverse mapping to Φ). We have

$$\text{Supp } V \subset \tilde{Q}. \tag{3.9}$$

Obviously the mapping Φ is a regular analytic one-to-one mapping of $\tilde{Q}$ onto $Q(3/2\ \rho_0)$. Now $V \in H^k_{PER}(Q(\pi))$, is symmetric with respect to the lines $\xi_i = \pm\ \pi/2$ and using (3.6) and (3.9) we obtain

$$||V||_{k,Q(\pi)} \leq C\ ||u||_{k,\Omega_0}\ . \tag{3.10}$$

It is well known that the partial sum t_p of the Fourier series of V gives the sequence of trigonometric polynomials $t_p \in P_p(Q(\pi))$ such that for $k \geq \ell$

$$||V-t_p||_{\ell,Q(\pi)} \leq Cp^{-(k-\ell)}||V||_{k,Q(\pi)}$$

$$\leq Cp^{-(k-\ell)}\|u\|_{k,\Omega}. \tag{3.11}$$

t_p are obviously symmetric with respect to the lines $\xi_i = \pm\ \pi/2$ as V is. It is readily seen that $t_p(\xi) = z_p(\Phi(\xi))$, where z_p is an algebraic polynomial of degree not higher than p. Because Φ is a regular, analytic mapping of $\tilde{Q}$ onto $Q(3/2\ \rho_0)$ (3.11) yields (3.5) for k,ℓ integral.

Now let us generalize our result to ℓ,k not integral. Recall that for given (fixed) p the polynomial z_p was constructed from a linear map L_p, $L_p u = z_p$, where L_p is a linear mapping of $H^0(\Omega_0)$ into $P_p(\Omega_0)$ satisfying (3.5) for ℓ,k integral. Applying general interpolation theory we get (3.5) for all $0 \leq \ell \leq k$.

The proof of the next theorem is more complicated.

THEOREM 3.2. Let $u \in H^k(\Omega_0) \cap H_0^1(\Omega_0)$. There exists a sequence $z_p \in P_{p,0}^{[S]}(\Omega_0)$ $p = 1,2,\ldots$ such that for any $k > 1$ (not necessarily integral) and any $\varepsilon > 0$

$$\|u-z_p\|_{1,\Omega_0} \leq Cp^{-(k-1)+\varepsilon}\|u\|_{k,\Omega_0} \tag{3.12}$$

where C is independent of p and u (it depends e.g. on ε and k).

Remark 1. In contrast to Theorem 3.1 the statement is false if only $P_{p,0}(\Omega_0)$ instead of $P_{p,0}^{[S]}(\Omega_0)$ is considered. This is easy to see if Ω_0 is e.g. an L-shaped domain as shown in Fig. 3.1.

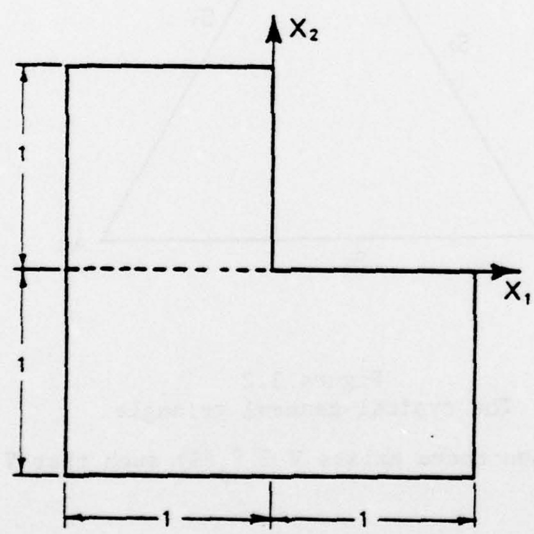


Figure 3.1
An L-shaped domain

In fact any $u \in P_{p,0}(\Omega_0)$ is zero in $(x_1,0)$, $0 < x_1 < 1$ and therefore - because it is a polynomial - has to be zero on the entire line $(x_1,0)$, $-1 < x_1 < 1$. This of course leads to a contradiction because of Sobolev's imbedding theorem of $H_0^1(\Omega_0)$ into $H^0(I(1))$.

Remark 2. It is not clear whether the term ε in (3.12) can be removed.

Remark 3. The theorem can be stated more generally. We have restricted ourselves to this case (i.e. $\|\cdot\|_{1,\Omega}$) only because it is sufficient for our purpose.

Before proving theorem 3.2 we will state a lemma.

Lemma 3.1. Let S be a triangle with vertices A_i, and sides s_i, $i = 1,2,3$ (see Fig. 3.2)

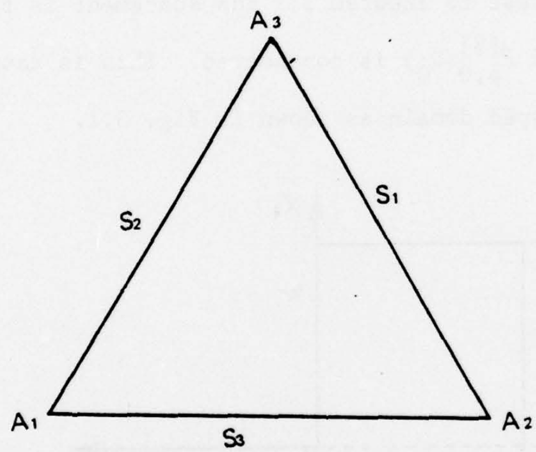


Figure 3.2
The typical general triangle

Let $v \in \mathcal{P}_{p,0}(s_1)$. Then there exists $V \in \mathcal{P}_p(S)$ such that $V = 0$ on s_2 and s_3, $V = v$ on s_1 and

$$||V||_{1,S} \leq C\ ||v||_{1,\, s_1} \tag{3.13}$$

where C (dependent on S) is independent of v and p.

Remark. By $v \in \mathcal{P}_{p,0}(s_1)$ we mean of course a polynomial in the variable s_1 so that $v = 0$ at the end points of s_1, the vertices A_2, A_3.

Proof. Without any loss of generality we can assume that S is the triangle shown in Fig. 3.3 with vertices (0,0) (1,0) (1,1)

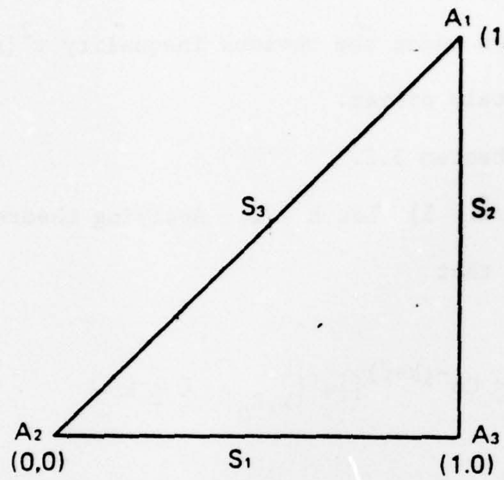


Figure 3.3
The Standard Triangle

Then $s_1 = (x_1, 0), 0 < x_1 < 1$. Because $v(x)$ is a polynomial and because $v(0) = v(1) = 0$ by assumption we have

$$v(x_1) = x_1(1-x_1)v_1(x_1).$$

with $v_1(x_1)$ a polynoimal of degree at most p-2. Define

$$V(x) = V(x_1,x_2) = v(x_1)\,\frac{(x_1-x_2)}{x_1} \tag{3.14}$$

Obviously $V \in P_p(S)$, $V = 0$ on s_2, s_3, and $V = v$ on s_1. Finally because $\frac{x_2}{x_1}$ is bounded on S we get $||V||_{0,S} \leq ||v||_{1,\,s_1}$. Writing

$$\int_S \left(\frac{\partial V}{\partial x_i}\right)^2 dx_1 dx_2 = \int_0^1 dx_2 \int_{x_2}^1 \left(\frac{\partial V}{\partial x_i}\right)^2 dx_1$$

we easily get (3.13) when using the obvious inequality $v^2(x_1) \leq x_1 ||v||^2_{1,s_1}$ and the lemma is completely proven.

Now we can prove Theorem 3.2.

Proof of Theorem 3.2. 1) Let $k > 2$. Applying theorem 3.1 there exists $z_p \in \mathcal{P}_p(\Omega_0)$ such that

$$||u-z_p||_{\ell,\Omega_0} \leq Cp^{-(k-\ell)}||u||_{k,\Omega_0}, \quad \ell \leq k . \tag{3.15}$$

Let $\{x^{[j]}\}$ $j = 1,\dots\hat{m}$ be the set of all vertices of the triangles $T_i \in \mathcal{S}$ belonging to $\partial\Omega$. Because $H^{1+\varepsilon}(\Omega_0)$, $\varepsilon > 0$ is imbedded in the space of continuous functions, we can obviously modify z_p to z_p^* by subtracting a polynomial of fixed degree $p_0 \leq \hat{m}$ so that

$$||u-z_p^*||_{\ell,\Omega_0} \leq C(p^{-(k-\ell)}+p^{-(k-1-\varepsilon)})||u||_{k,\Omega_0} \tag{3.16}$$

with $\varepsilon > 0$ arbitrary. In fact $\tilde{z} = z_p - z_p^*$ is a polynomial of fixed (independent of p) degree $\hat{m}$ determined by its values at the points $\{x^{[j]}\}$. By theorem 3.1 we have $|z_p(x^{[j]})| \leq Cp^{-(k-1)+\varepsilon}||u||_{k,\Omega_0}$ and so $||\tilde{z}||_{\ell,\Omega_0} \leq Cp^{-(k-1)+\varepsilon}$ for all $0 \leq \ell \leq \hat{m}$. Because obviously $||\tilde{z}||_{r,\Omega_0} = ||\tilde{z}||_{\tilde{m},\Omega_0}$ for any $r \geq \tilde{m}$, (3.16) follows readily.

We see that on every side $s \subset \partial\Omega$ of $T \in \mathcal{S}$ we have $u = 0$ and therefore $||z_p^*||_{1,s} \leq ||u-z_p^*||_{2,\Omega_0}$ by applying the Sobolev imbedding theorem. Using lemma 3.1 we can now find $z_p^{**} \in \mathcal{P}_p^{[S]}(\Omega_0)$ so that $z_p^0 = z_p^* - z_p^{**} \in \mathcal{P}_{p,0}^{[S]}(\Omega_0)$ and

$$||u-z_p^0||_{1,\Omega_0} \leq C||u-z_p^*||_{2,\Omega_0} \leq Cp^{-(k-2)}||u||_{k,\Omega_0} \tag{3.17}$$

where C depends on k > 2 but is independent of u. (3.17) can obviously be written as

$$||u-z_p^0||_{1,\Omega_0} \leq Cp^{-(k-1)(1-\frac{1}{k-1})}||u||_{k,\Omega_0}. \tag{3.18}$$

2) Let now R_p be the orthogonal projection in the scalar product of H^1 of

$$H^k(\Omega_0) \cap H_0^1(\Omega_0) \text{ onto } P_{p,0}^{[S]}(\Omega_0).$$

Let $z_p = R_p u$. We obviously have

$$||z_p-u||_{1,\Omega_0} \leq ||u||_{1,\Omega_0} \tag{3.19}$$

and from (3.18) for k > 2 we have

$$||z_p-u||_{1,\Omega_0} \leq C(k)p^{-(k-1)(1-\frac{1}{k-1})}||u||_{k,\Omega_0}. \tag{3.20}$$

For 1 < s < k, let

$$\tilde{H}^s(\Omega_0) = [H_0^1, H^k(\Omega_0) \cap H_0^1]_{\frac{s-1}{k-1}, 2}.$$

We obtain by applying interpolation theory

$$||z_p-u||_{1,\Omega_0} \le C(k,s)\ p^{-\mu}||u||_{\tilde{H}^s(\Omega_0)} \tag{3.22}$$

with

$$\mu = (k-1)(1-\frac{1}{k-1})(\frac{s-1}{k-1}) = (s-1)(1-\frac{1}{k-1})\ . \tag{3.23}$$

Therefore given $\varepsilon > 0$ and $s > 1$ we can select k_0 so that

$$(s-1)(1-\frac{1}{k_0-1}) \ge (s-1) - \varepsilon' \tag{3.24}$$

and so (3.12) holds when the norm $||u||_{\tilde{H}^k(\Omega_0)}$, instead of $||u||_{k,\Omega_0}$, is used.

On the other hand from [3], see also [29], it follows that when Ω_0 is a polygon, then the spaces $\tilde{H}^k(\Omega_0)$ and $H^k(\Omega_0) \cap H_0^1$ are equivalent. This completes the proof.

3.4 The inverse approximation theorem

We have proven theorems about approximability properties of the spaces

$$P_p^{[S]}(\Omega_0) \quad \text{and} \quad P_{p,0}^{[S]}(\Omega_0)\ .$$

Now we will prove the inverse approximation theorem.

THEOREM 3.3. Let $v \in H^k(Q(\rho))$ and let there exists a sequence of polynomials $z_p \in P_p(Q(\rho))$, $p = 1,2,\ldots$ such that

$$||v-z_p||_{k,Q(\rho)} \le \frac{K}{p^r} \qquad r > 0 \tag{3.25}$$

$k \geq 0$ integral. Then $v \in H^{k+r-\varepsilon}(Q(\rho^*))$, $\rho^* < \rho$, $\varepsilon > 0$ arbitrary and after restriction of v onto $Q(\rho^*)$

$$||v||_{k+r-\varepsilon,Q(P^*)} \leq A(\rho,\rho^*,k,r,\varepsilon)\ [||v||_{0,Q(\rho)} + K] \tag{3.26}$$

Proof. Let $\omega = (x_1^2-\rho^2)^{k+2}(x_2^2-\rho^2)^{k+2}$. Then writing $v^* = v\omega$, $z^*_{p+4(k+2)} = z_p\omega$

$$V^*(\xi) = v^*(\Phi(\xi))$$

$$t^*_{p+4(k+2)}(\xi) = z^*_{p+4(k+2)}(\Phi(\xi)) \in F_{p+4(k+2)}(Q(\pi))$$

with

$$\Phi(\xi) = x$$

$$x_i = \rho \sin \xi_i$$

we obtain

$$||V^*-t^*_{p+4k+2}(\xi)||_{k,Q(\pi)} \leq \frac{CK}{p^r}\ . \tag{3.27}$$

Now using theorem 5.4.1 p. 200 of [17] [For a proof using the basic interpolation theory directly see e.g. [3]] it follows that

$$||V^*||_{k+r-\varepsilon,Q(\pi)} \leq A[||V^*||_{k,Q(\pi)} + K]$$

(More precisely by the mentioned theorems we obtain the norm of V* in the Nikolsky spaces $B^{k+r}_{2,\infty}(Q(\pi)$ which majorizes the norm $\|\ \|_{k+r-\varepsilon,Q(\pi)})$.

Now using the fact that the mapping Φ is a one-to-one analytic one on $Q(\rho^*)$ and $\omega(x) \geq \alpha > 0$ on $Q(\rho^*)$ we immediately obtain (3.26). Inequality (3.25) holds only on $Q(\rho^*)$ and in general is not true on $Q(\rho)$. Nevertheless we can prove the next theorem.

THEOREM 3.4. Let $v \in H^k(Q(\rho))$ and suppose that the other assumptions of Theorem 3.3 are satisfied, then $v \in H^{k+r/2-\varepsilon}(Q(\rho))$ and

$$\|v\|_{k+r/2-\varepsilon,Q(\rho)} \leq A(\varepsilon)[\|v\|_{k,Q(\rho)} + K] \tag{3.28}$$

The proof of this theorem is a consequence of the above mentioned theorem 5.4.1 in [17] provided that for integral $k \geq 0$ the following inequality of Bernstein type

$$\|z_p\|_{k,Q(\rho)} \leq Cp^{2k}\|z_p\|_{0,Q(\rho)} \tag{3.29}$$

holds for any $z_p \in P_p(Q(\rho))$ with C independent of p and z_p.

Let us remark that in the case of trigonometrical polynomials we have in (3.29) the term p^k instead p^{2k}. We will prove (3.29) in the next few lemmas.

Lemma 3.2. Let $z_p(x)$, $x \in I(1)$ be a polynomial (in one variable) of degree p. Then

$$\left\|\frac{d^s z_p}{dx_s}\right\|_{0,I} \leq C(s)p^{2s}\|z_p\|_{0,I}$$

Proof. By Schmidt's inequality we have

$$\int_{-1}^{+1} f'(x)^2 dx \leq \frac{(N+1)^4}{2}\int_{-1}^{+1} f^2(x)dx \tag{3.31}$$

when f(x) is a polynomial of degree not higher than N. (See [6]), (3.31) then yields (3.30) easily.

The next lemma follows easily from the previous lemma.

Lemma 3.3. Let $z_p(x) \in \mathcal{P}_p(Q(1))$. Then for any integral $k \geq 0$

$$||z_p||_{k,Q(1)} \leq C(k)p^{2k}||z_p||_{0,Q(1)} \tag{3.32}$$

Proof. For every fixed x_2 we have $z_p(x_1, x_2) \in \mathcal{P}_p(I)$ and therefore using Lemma 3.2 we get

$$\int_{-1}^{+1}[\frac{\partial z_p}{\partial x_1}(x_1,x_2)]^2 dx_1 \leq Cp^4\int_{-1}^{+1} z_p^2(x_1,x_2)dx_1 \tag{3.33}$$

Integrating (3.33) with respect to x_2 we obtain

$$\left|\left|\frac{\partial z_p}{\partial x_1}\right|\right|_{0,Q(1)} \leq Cp^2||z_p||_{0,Q(1)} \tag{3.34}$$

and analogously for $\frac{\partial z_p}{\partial x_2}$. By induction we get (3.32). Obviously (3.32) is equivalent to (3.29) and therefore Theorem 3.4 is completely proven.

3.5 The convergence of the p-version of the finite element method

Theorems 3.1 and 3.2 lead immediately to an a priori estimate of the rate of convergence of the p-version of the finite element method.

THEOREM 3.5. Let $u_0 \in H^k(\Omega_0)$, $k > 1$ be the exact solution of the problem (3.1), (3.2) and let u_p be the finite element approximation then

$$||u_0-u_p||_{1,\Omega_0} \leq C(k,\varepsilon)p^{-(k-1)+\varepsilon}||u_0||_{k,\Omega} \tag{3.35}$$

when $\varepsilon > 0$ is arbitrary. For the boundary condition $\Gamma_u = \frac{\partial u}{\partial n}$, ε can be set equal to zero.

A polynomial of degree p has N degrees of freedom with $N \approx p^2$ therefore $P_p^{[S]}$ (and $P_{p,0}^{[S]}$) has dimension N with $N \approx p^2$ and (3.35) can be rewritten in the form

$$||u_0-u_p||_{1,\Omega_0} \leq C(k,\varepsilon)N^{-\frac{(k-1)}{2}+\varepsilon}||u_0||_{k,\Omega_0} . \tag{3.36}$$

For the conventional finite element (h-version) approach with quasi-uniform mesh we have

$$||u_0-u_h||_{1,\Omega_0} \leq C\, h^{\mu}||u_0||_{k,\Omega_0} \tag{3.37}$$

with $\mu = \min\,(k-1,\ q)$

where q is the degree of the complete polynomial used in the elements. Realizing that in this case the number of degrees of freedom N satisfies $N \approx h^{-2}$ we can rewrite (3.3.7) in another form

$$||u_0-u_h||_{1,\Omega_0} \leq C\, N^{-\mu/2}||u_0||_{k,\Omega_0} \tag{3.38}$$

and this rate of convergence is an optimal one (possibly up to $\bar{\varepsilon} > 0$ arbitrary) (see [3]). This shows that the p-version gives results which are (neglecting $\varepsilon > 0$) not worse than the conventional h-version with quasi-uniform mesh if we compare the number of degrees of freedom leading to the same accuracy. In addition the convergence can be much better because we do not have the restriction on the convergence rate due to the degree of the elements as we have in the usual h-version.

Further as will be seen in the next section (see theorem 4.3) under some conditions which are usually satisfied in practice the factor 1/2 in (3.36) can be removed and then the p-version will be superior in comparison to the usual (h-version) finite element method with quasi-uniform mesh.

Let us remark on the other hand that when the usual (h-version) with the proper refinement of elements is used then in general the convergence rate can be better than in the case of the p-version with fixed mesh - see [3]. Although the general theory for a method combining the h and p version in an obvious manner is not yet developed, we can expect that the theoretical and practical advantages of both approaches can be combined.

Let us assume now that the convergence rate of the p-version of the finite element method is r, i.e. assume that

$$||u_o-u_p||_{1,\Omega} \leq K\,p^{-r} \,. \tag{3.39}$$

Then the following theorem holds.

THEOREM 3.6. Let $u_0 \in H^1(\Omega)$ and assume that (3.39) holds. Then

i) $u_0 \in H^{1+r-\varepsilon}(\Omega^*)$ where Ω^* is any domain such that $\bar{\Omega}^* \subset T_i$, $i = 1,\dots m$ where T_i are the triangles of the triangulation S and

$$||u_0||_{1+r-\varepsilon,\Omega^*} \leq A(\Omega^*,r,\varepsilon)(||u_0||_{1,\Omega_0}+K) \tag{3.40}$$

ii) $u_0 \in H^{1+r/2-\varepsilon}(T_i)$, $i = 1,\dots m$

and

$$||u_0||_{1+r/2-\varepsilon,T_i} \leq A(T_i,r,\varepsilon)(||u_0||_{1,\Omega_0}+K) \tag{3.41}$$

Proof. Theorems 3.3 and 3.4 are obviously valid not only for a rectangle Q but for any parallelogram.

i) From theorem 3.3 we see that (3.42) holds for any Ω^* of the form of a parallelogram. This is obviously sufficient for (3.40) in general.

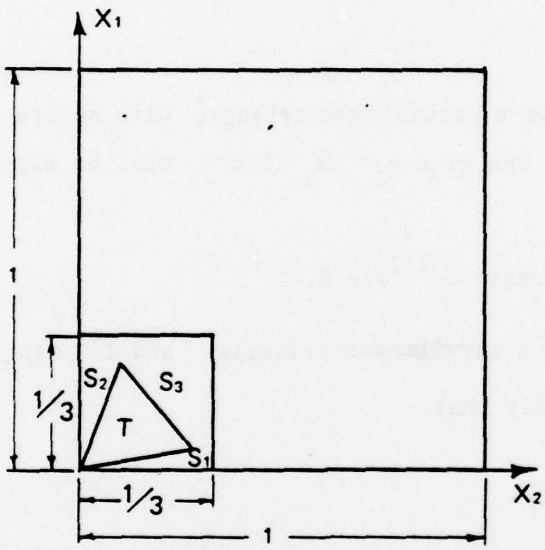
ii) Because any T_i can be covered (with overlapping) by a finite set of parallelograms (3.41) follows directly from Theorem 3.4.

The practical importance of Theorem 3.6 lies in the observation that the triangulation of Ω has to be made so that the possible singularities are located at the boundaries of T_i. Exactly this was done in the linear elastic fracture mechanics problems analyzed by Szabo and Mehta [26] using the p-version of the finite element method.

4. THE SINGULARITY PROBLEM AND THE P-VERSION OF THE FINITE ELEMENT METHOD

4.1 Preliminaries

In this section we will write $\hat{Q}$ instead of $\hat{Q}(1)$. $\hat{Q}(\rho)$ was defined in section 2 . Let T_1 be an open triangle with the vertex in the origin and $\overline{T}_1 \subset \hat{Q}\ (1/3) \cup (0,0)$. See figure 4.1.



4.1 Triangle with vertex at singularity

Denote the sides of T going through the origin by s_1 and s_2 and the remaining one by s_3.

By $\hat{\Phi}$ we denote the mapping (one to one) of $\hat{Q}(\pi/2)$ onto $\hat{Q}$, defined so that

$$\hat{\Phi}(\xi) = x, \qquad \xi \equiv (\xi_1, \xi_2) \in \hat{Q}(\pi/2)$$

$$x \equiv (x_1, x_2) \in \hat{Q}$$

with

$$x_i = \sin^2 \xi_i \qquad i = 1,2.$$

By $\hat{\Phi}^{-1}$ we denote the inverse mapping to $\hat{\Phi}$. Further let

$$T^{\Phi} = \hat{\Phi}^{-1}(T), \qquad s_i^{\Phi} = \hat{\Phi}^{-1}(s_i)$$

$i = 1,2,3.$

T^{Φ} now will be a curvilinear triangle with smooth sides and positive angles. In fact the line $x_2 = cx_1$ $(0<c<\infty)$ will be mapped into $\sin^2\xi_2 = c \sin^2\xi_1$ and so

$$\xi_2 = \arcsin c^{1/2} \sin \xi_1.$$

Therefore, T^{Φ} is a curvilinear triangle and $\overline{T}^{\Phi} \subset \hat{Q}(\rho_0) \cup (0,0)$, $\rho_0 = \arcsin \frac{1}{\sqrt{3}}$.

We see also readily that

$$\frac{\sin 2\xi_1}{\sin 2\xi_2}$$

is a function bounded from above and below on T^{Φ}.

Now let $v \in H^1(T)$ be given and define

$$V(\xi) = v(\hat{\Phi}(\xi)).$$

We prove

Lemma 4.1. Let $v \in H^1(T)$ then $V \in H^1(T^{\Phi})$ and

$$c_1 \|v\|_{1,T} \le \|V\|_{1,T^{\Phi}} \le c_2 \|v\|_{1,T} \tag{4.1}$$

with $0 < c_1 < c_2 < \infty$ independent of v.

Proof. First let us show that

$$c_1 \left\| \frac{\partial v}{\partial x_i} \right\|_{0,T} \leq \left\| \frac{\partial V}{\partial \xi_i} \right\|_{0,T^\Phi} \leq c_2 \left\| \frac{\partial v}{\partial x_i} \right\|_{0,T}. \tag{4.2}$$

We have

$$\frac{\partial V}{\partial \xi_i}(\xi) = \frac{\partial v}{\partial x_i}\frac{\partial x_i}{\partial \xi_i} = \frac{\partial v}{\partial x_i}(\hat{\Phi}(\xi)) \sin 2\xi_1 .$$

Therefore,

$$\int_{T^\Phi} \left(\frac{\partial V}{\partial \xi_i}\right)^2 d\xi_1 d\xi_2 = \int_T \left(\frac{\partial v}{\partial x_i}\right)^2 \sin^2 2\xi_1 \frac{dx_1}{\sin 2\xi_1} \frac{dx_2}{\sin 2\xi_2} .$$

Because as we mentioned $\dfrac{\sin 2\xi_1}{\sin 2\xi_2}$ is a function bounded from above and below, we get (4.2).

Further we have,

$$\int_{T^\Phi} V^2 d\xi = \int_T v^2 \frac{dx_1}{\sin 2\xi_1} \frac{dx_2}{\sin 2\xi_2}$$

$$\leq \left[\int_T v^{2p} dx\right]^{\frac{1}{p}} \left[\int_T \left(\frac{1}{\sin 2\xi_1} \frac{1}{\sin 2\xi_2}\right)^q dx\right]^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{4.3}$$

Because in the neighborhood of the origin we have $\xi_i \approx x_i^{1/2}$ it follows that $\dfrac{1}{\sin 2\xi_1} \approx x_i^{-1/2}$ and therefore for p=3 and q=3/2, the second term in (4.3) is bounded. On the other hand by the Sobolev imbedding theorem we have

$$\left| \int_T v^6 dx \right|^{1/3} \leq C \| v \|_{1,T}^2$$

and so we get

$$\left[\int_{T^\Phi} V^2 d\xi \right]^{1/2} \leq C \| v \|_{1,T}. \tag{4.4}$$

Now (4.4) together with (4.2) gives

$$\| V \|_{1,T^\Phi} \leq C \| v \|_{1,T}.$$

(4.3) yields also

$$\| v\|_{0,T} \leq \| V\|_{0,T^{\Phi}}$$

and we easily complete the proof of the inequality (4.1)

Lemma 4.2. Let $v(x)$ be defined on $\hat{I}(1)$ (as a function of one variable $0<x<1$) and let

$$\int_0^1 v^2 x^{-1}\, dx + \int_0^1 \left(\frac{dv}{dx}\right)^2 x\, dx \leq A^2 < \infty. \tag{4.5}$$

Let S be the triangle with vertices (0,0), (1,0), (1,1) (as in Figure 3.3) and let

$$u(x_1,x_2) = v(x_1)\left(1 - \frac{x_2}{x_1}\right).$$

Then

$$\| u\|_{1,S} \leq CA \tag{4.6}$$

with C independent of v.

Proof. We have

$$\begin{aligned} \int_S u^2 dx &= \int_0^1 v^2(x_1)\, dx_1 \int_0^{x_1} \left(1 - \frac{x_2}{x_1}\right)^2 dx_2 \\ &\leq 2\int_0^1 v^2(x_1)\, dx_1 \left[x_1 + \frac{1}{3} x_1\right] \leq CA^2 \end{aligned} \tag{4.7}$$

Further

$$\int_S \left(\frac{\partial u}{\partial x_1}\right)^2 dx \le 2\left[\int_0^1 \left(\frac{\partial v}{\partial x_1}\right)^2 dx_1 \int_0^{x_1} \left(1-\frac{x_2}{x_1}\right)^2 dx_2 + \int_0^1 v^2\, dx_1 \int_0^{x_1} \frac{x_2^2}{x_1^4}\, dx_2\right]$$

$$\le 2\left[\int_0^1 x_1 \left(\frac{\partial v}{\partial x_1}\right)^2 dx_1 + \int_0^1 x_1^{-1} v^2\, dx_1\right] \le CA^2 \tag{4.8}$$

$$\int_S \left(\frac{\partial u}{\partial x_1}\right)^2 dx \le \int_0^1 v^2 dx_1 \int_0^{x_1} \frac{1}{x_1^2}\, dx_2$$

$$\le \int_0^1 v^2\, x_1^{-1}\, dx_1 \le A^2 \tag{4.9}$$

Combining (4.7) (4.8) (4.9) we get (4.6) and the lemma is proven.

Remark. If v is a smooth function (4.5) implies that $v(0) = 0$. In addition if v is a polynomial then u is a polynomial (in two variables) too.

4.2 Approximation Properties of the space $P_p(T)$

Let us introduce a one parameter family $\Psi_\gamma(\Delta)$ $0<\Delta<\Delta_0$ ($\gamma>0$, fixed) of functions defined on $\hat{Q}$. A function $u_\Delta(x) \in \Psi_\gamma(\Delta)$ iff

i) $u_\Delta \in E(\hat{Q})$ (not $E(\bar{\hat{Q}})$)

ii) $\text{Supp } u_\Delta \subset R_\alpha$, $\alpha > 1$

where we define

$R_\alpha \equiv \{x \in \hat{Q}(1/3),\ \frac{x_1}{\alpha} < x_2 < \alpha x_1\}$

iii) $|\mathcal{D}^k u_\Delta| \le C(|k|)\,]x[^{-\{|k|-\gamma\}}$

for $]x[> \Delta$ and any $k \equiv (k_1, k_2)$, $k_i \ge 0$ integral with

$]x[= \min(x_1, x_2)$

$$\{a\} = \begin{cases} a \text{ for } a \ge 0 \\ 0 \text{ for } a < 0 \end{cases}$$

iv) $|\mathcal{D}^k u_\Delta| \leq C(|k|)\ \Delta^{-\{|k|-\gamma\}}$ and $C(k)$ is independent of Δ.

Denote now as before

$$U_\Delta(\xi) = u_\Delta(\hat{\Phi}(\xi))$$

and

$$\Psi_\gamma^\Phi(\Delta) \equiv \{U_\Delta(\xi) \mid u_\Delta \in \Psi_\gamma(\Delta)\}.$$

Now we prove the following theorem:

THEOREM 4.1

Let

$$U_\Delta(\xi) \in \Psi_\gamma^\Phi(\Delta)$$

Then

$$U_\Delta \in H^k(\hat{Q}(\pi/2)) \text{ for any } k \geq 0$$

and

$$\| U_\Delta \|_{k,Q(\pi/2)} \leq C(k)\Delta^{-\{1/2\{k-2\gamma\}-1/2\}} \tag{4.10}$$

with C independent of Δ.

First we introduce some auxiliary lemmas.

Lemma 4.3. For $0<t<\pi/2$, and $n \geq 1$, $1 \leq k \leq n$, k,n integral define

$${}^n W_k(t) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \sin^{2(k-j)} t \frac{d^n}{dt^n} \sin^{2j} t \tag{4.11}$$

then

$$|{}^n W_k(t)| \leq C(n)\ t^{\{2k-n\}} \tag{4.12}$$

Proof. Obviously ${}^n W_k(t)$ is a trigonometric polynomial. In the neighborhood of $t=0$ we have

$$\sin^{2k} t = t^{2k} + 0\,(t^{2k+2})$$

and therefore,

$$\left| \frac{d^n \sin^{2k}}{dt^n}(t) \right| \leq C(n) t^{\{2k-n\}}. \tag{4.13}$$

Hence for $j \leq k \leq n$

$$\sin^{2(k-j)} t \; \frac{d^n}{dt^n} \sin^{2j}(t) \leq C(n) t^{2(k-j)} \, t^{\{2j-n\}}$$

$$\leq C(n) \, t^{A_1(k,j,n)}$$

where

$$A_1(k,j,n) = 2(k-j) + \{2j-n\}. \tag{4.14}$$

It is easy to check that

$$A_1(k,j,n) \geq \{2k-n\}$$

and this yields (4.12).

Lemma. 4.4

Let

$$U_\Delta \in \Psi_\gamma^\phi (\Delta)$$

Then

$$\left| \frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1} \partial \xi_2^{k_2}} (\xi) \right| \leq C(|k|) \,]\xi[^{\,-\{|k|-2\gamma\}} \tag{4.15}$$

for $]\xi[\; \geq \theta$, $\theta = \arcsin \Delta^{1/2}$.

Proof. We have (see e.g. [9], p. 19)

$$\frac{\partial^{k_1} U_\Delta}{\partial \xi_1^{k_1}} = \sum_{j=1}^{k_1} W_j(\xi_1) \frac{1}{j!} \frac{\partial^j u_\Delta}{\partial x_1^j} (\hat{\Phi}(\xi))$$

and therefore

$$\frac{\partial^{|k|}U_\Delta}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}} = \sum_{j=1}^{k_1}\sum_{\ell=1}^{k_2} {}^{k_1}W_j(\xi_1)\,{}^{k_2}W_\ell(\xi_2)\frac{1}{j!\ell!}\,\frac{\partial^{j+\ell}u_\Delta}{\partial x_1^{j}\partial x_2^{\ell}}(\hat{\Phi}(\xi)) \tag{4.16}$$

Using lemma 3.3 we get

$$\left|\frac{\partial^{|k|}U_\Delta}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}\right| \leq C(k_1,k_2)\sum_{j=1}^{k_1}\sum_{\ell=1}^{k_2} \xi_1^{\{2j-k_1\}}\xi_2^{\{2\ell-k_1\}}\,]\hat{\Phi}(\xi)[^{-\{j+\ell-\gamma\}} \tag{4.17}$$

with

$$]\hat{\Phi}(\xi)[> \Delta .$$

Because

$$-C]\xi[^2 \geq]\hat{\Phi}(\xi)[\geq C]\xi[^2$$

(4.17) can be written in the form

$$\left|\frac{\partial^{|k|}U_\Delta}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}\right| \leq C(k_1,k_2)\sum_{j=1}^{k_1}\sum_{\ell=1}^{k_2} \xi_1^{\{2j-k_1\}}\xi_2^{\{2\ell-k_2\}}\,]\xi[^{-2\{j+\ell-\gamma\}} \tag{4.18}$$

with $]\xi[\geq \theta$

and therefore

$$\left|\frac{\partial^{|k|}U_\Delta}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}\right| \leq C(k_1,k_2)\sum_{j=1}^{k_1}\sum_{\ell=1}^{k_2}]\xi[^{A_2(j,\ell,k_1,k_2,\gamma)} \tag{4.19}$$

with

$$A_2(j,\ell,k_1,k_2,\gamma) = \{2j-k_1\} + \{2\ell-k_2\} - 2\{j+\ell-\gamma\} \tag{4.20}$$

Here we used the fact that for $\xi \in \operatorname{supp} U_\Delta$ we have

$$0 < c_1 \leq \frac{\xi_1}{\xi_2} \leq c_2 < \infty$$

By simple computation we get

$$A_2 \geq -\{|k|-2\gamma\} \tag{4.21}$$

and now (4.15) follows from (4.19) and (4.21).

Now we will prove Theorem 4.1.

Proof of Theorem 4.1. Define

$$R_\alpha^\Phi = \hat{\Phi}^{-1}\ [R_\alpha]$$

Because by assumption supp $U_\Delta \subset R_\alpha^\Phi$, we have

$$\| U_\Delta\|^2_{k,Q(\pi/2)} \leq \| U_\Delta\|^2_{k,Q(\theta)\ \cap R_\alpha^\Phi} + \| U_\Delta\|^2_{k,\ R_\alpha^\Phi - Q(\theta)} \tag{4.22}$$

Now we will separately estimate the terms in (4.22).

On $Q(\theta)$ using (4.16), lemma 4.3, and the property (iv) of u_Δ

$$\left|\frac{\partial^{|m|} U_\Delta}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}}\right|^2 \leq C(m_1,\ m_2) \times \sum_{j=1}^{m_1}\sum_{\ell=1}^{m_2} \xi_1^{2\{2j-m_1\}}\ \xi_2^{2\{2\ell-m_2\}}\ \Delta^{-2\{j+\ell-\gamma\}}$$

Because $\theta \leq C\Delta^{1/2}$ we get

$$\left\| \frac{\partial^{|m|} U_\Delta}{\partial \xi_1^{m_1} \partial \xi_2^{m_2}} \right\|^2_{0,Q(\theta)}$$

$$\leq C(m_1,m_2) \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \Delta^{-2\{j+\ell-\gamma\} + 1/2[2\{2j-m_1\} + 2\{2\ell-m_2\}] + 1}$$

$$\leq C(m_1,m_2) \sum_{j=1}^{m_1} \sum_{\ell=1}^{m_2} \Delta^{A_2(j,\ell,m_1,m_2,\gamma) + 1}$$

$$\leq C(m_1,m_2)\, \Delta^{-\{m-2\gamma\}+1} . \tag{4.23}$$

On $R^\Phi_\alpha - Q(\theta)$ using lemma 4.4, we have

$$\left\| \frac{\partial^{|m|} U_\Delta}{\partial \xi_1^{m_1} \partial \xi_2^{m_2}} \right\|^2_{0,R^\Phi_2 - Q(\theta)}$$

$$\leq C(m_1,m_2) \int_{C\Delta^{1/2}}^{\pi/2} \xi_1 \, \xi_1^{-2\{|m|-2\gamma\}} \, d\xi_1$$

$$\leq C(m_1, m_2)\, (\Delta^{1/2})^{-\{2\{|m|-2\gamma\}-2\}} . \tag{4.24}$$

(4.23) and (4.24) yield (4.10) .

THEOREM 4.2. Let $u_\Delta \in \Psi_\gamma(\Delta)$, be continuous on $\bar{\hat{Q}}$, $u_\Delta(0,0) = 0$, and $u = 0$ on the side s_3 of T. Then there exists $z_p \in \mathcal{P}_p(T)$ such that for any $k \geq 2\gamma+1$, k integral

i). $\| u_\Delta - z_p \|_{1,T} \leq C(k) p^{-(k-2)} \Delta^{-1/2 \{k-2\gamma\}-1/2}$ (4.25)

ii) $z_p = 0$ on the side s_3 of T

iii) $$\left[\int_{s_i} (u_\Delta - z_p)^2 \, \mathscr{s}_i^{-1} \, d\mathscr{s}_i + \int_{s_i} \left(\frac{d}{d\mathscr{s}_i}(u_\Delta - z_p)\right)^2 \mathscr{s}_i \, d\mathscr{s}_i\right]^{1/2}$$

$$\leq C(k)p^{-(k-2)} \Delta^{-1/2\{k-2\gamma\}+1/2}, \; i=1,2 \qquad (4.26)$$

where we denote by $\mathscr{s}_i$ the length parameter of the side s_i measured from the origin.

Proof. By theorem 4.1 the function U_Δ satisfies (4.10). Therefore there exists a sequence of trigonometrical polynomials with period π and symmetric with respect to the lines $\xi_1 = \pm\pi/2$, $\xi_1 = 0$, $\xi_2 = \pm\pi/2$, $\xi_2 = 0$ such that for $0 \leq m < k$

$$\| U_\Delta - t_p \|_{m, \hat{Q}(\pi/2)} \leq C(m,k)p^{-(k-m)} \Delta^{-1/2\{k-2\gamma\}+1/2}. \qquad (4.27)$$

Recalling that $U_\Delta = 0$ at the vertices of T^Φ we can modify t_p so that $t_p = 0$ at these vertices and (4.27) holds for all $m > 1 + \varepsilon$. In addition using the trace theorem we have

$$\| t_p - U_\Delta \|_{1, s_i^\Phi} \leq C(k)p^{-(k-2)} \Delta^{-1/2\{k-2\gamma\}+1/2} \quad i=1,2,3. \qquad (4.28)$$

Defining $z_p\,(\Phi(\xi)) = t_p(\xi)$, z_p is an algebraic polynomial of degree p. Using lemma 4.1 we get (4.25). By assumption $t_p = 0$ and $U_\Delta = 0$ at the vertices of T^Φ and therefore $z_p = 0$ at the vertices of T. Further we have

$$|U_\Delta - t_p(\mathscr{s}_i^\Phi)| < C(\mathscr{s}_i^\Phi)^{1/2} \, \| t_p - U_\Delta \|_{1, s_i^\Phi}$$

where we have denoted the length parameter of s_i^Φ, $i = 1, 2$ measured from the origin by $\mathscr{s}_i^\Phi$. Therefore for $z_p(\mathscr{s}_i)$ on s_i we get for $i = 1,2$

$$\int_{s_i} (z_p-u)^2 s_i^{-1}\, ds_i \le C \int_{s_i^\Phi} (t_p-U_\Delta)^2 (s_i^\Phi)^{-2}\, s_i^\Phi\, d s_i^\Phi$$

$$\le C \int_{s_i^\Phi} (t_p-U_\Delta)^2 (s_i^\Phi)^{-1}\, ds_i^\Phi$$

$$\le C \,\| t_p-U_\Delta\|^2_{1,\, s_i^\Phi}. \tag{4.29}$$

Further,

$$\int_{s_i} \left(\frac{\partial}{\partial s_i}(z_p-u_\Delta)\right)^2 s_i\, ds_i \le C \int_{s_i^\Phi} \left(\frac{\partial}{\partial s_i^\Phi}(t_p-U_\Delta)\right)^2 s_i^\Phi\, ds_i^\Phi$$

$$\le \| t_p-U_\Delta\|^2_{1,\, s_i^\Phi} \tag{4.30}$$

and therefore combining (4.29),(4.30) and (4.28) we get (4.26).

Realizing that on s_3 the mapping $\hat{\Phi}$ is an analytic one we can use lemma (3.1) and achieve $z_p = 0$ on s_3. So theorem 4.2 is completely proven.

Remark to theorem 4.2. We have assumed that triangle T is situated as in Fig. 4.1. It is easy to see that a linear transformation of the coordinates does not make any change in the theorem. Therefore, theorem 4.2 is true for any triangle T with vertex at the origin.

4.3. A concrete family $\Psi_\gamma(\Delta)$.

First denote by $\chi(x)$ $0<x<\infty$ a function with all continuous derivatives such that $\chi(x) = 0$ for $0 < x < 1/2$, and $\chi(x) = 1$ for $1<x<\infty$. Further let $\chi_\Delta(x) = \chi(\frac{x}{\Delta})$.

Let now $u \in \hat{Q}$

$$u = \rho(r)\ \Theta(\phi) \tag{4.31}$$

be given with r, ϕ being the polar coordinates and $\Theta(\phi)$ be a function with all continuous derivatives. Further assume that u has support in $\overline{R}_\alpha$ and $u = 0$ on s_3. In addition let $\rho(r)$ be continuous $\rho(0) = 0$ and

$$\left|\frac{d^n\rho}{dr^n}\right| \leq r^{\gamma-n}\, C(n) \qquad \text{with } \gamma > 0 \tag{4.32}$$

Now let $\rho_\Delta = \chi_\Delta(r)\rho(r)$ and $u_\Delta = \rho_\Delta\Theta(\phi)$. Then obviously $u_\Delta = 0$ on s_3 and has compact support in $\overline{R}_\alpha$ for all Δ.

Let us show now that u_Δ is a $\Psi_\gamma(\Delta)$ family of functions. Obviously, parts i) and ii) of definition of $\Psi_\gamma(\Delta)$ are satisfied.

Further

$$\left|\frac{\partial^k u_\Delta}{\partial r^k}\right| \leq C \sum_{j=0}^{k} \left|\frac{\partial^j \chi_\Delta}{\partial r^j}\right| \left|\frac{\partial^{k-j}\rho}{\partial r^{k-j}}\right| \leq C \sum_{j=0}^{k} \Delta^{-j}\ \Delta^{\gamma-k+j} \leq C(k)\Delta^{\gamma-k} \tag{4.33}$$

because of (4.32) and the fact that $\chi_\Delta = 0$ for $r \leq \Delta/2$,

and

$$\left|\frac{\partial^{|k|}u_\Delta}{\partial x_1^{k_1}\partial x_2^{k_2}}\right| \leq C \sum_{\ell=0}^{|k|} \frac{\partial^\ell \chi_\Delta \rho}{\partial r^\ell}\ \frac{1}{r^{|k|-\ell}}$$

$$\leq C \sum_{\ell=0}^{|k|}\sum_{j=0}^{\ell} \left|\frac{\partial^j \chi_\Delta}{\partial r^j}\right| \left|\frac{\partial^{\ell-j}\rho}{\partial r^{\ell-j}}\right| \frac{1}{r^{|k|-\ell}}$$

$$\leq C \sum_{\ell=0}^{|k|} \Delta^{\gamma-\ell}\ \Delta^{-|k|+\ell} \leq C(|k|)\Delta^{\gamma-|k|} \tag{4.34}$$

so property iv) of the $\Psi_\gamma(\Delta)$ family is satisfied.

Because $\chi_\Delta(r) = 1$ for $r \geq \Delta$, (4.31) yields the property iii) of the $\Psi_\gamma(\Delta)$ family.

Let us now show another property of u_Δ in our concrete case.

Lemma 4.5. Let u be given by (4.31) with ρ and Θ satisfying the conditions spelled out above and let $u_\Delta = \chi_\Delta\ (r)\rho(r)\Theta(\phi)$. Then

$$\| u-u_\Delta \|_{1,\hat{Q}} \leq C\ \Delta^\gamma \tag{4.35}$$

with C independent of Δ.

Proof. We have $v = u - u_\Delta = (1-\chi_\Delta)\rho(r)\Theta(\phi)$ and therefore we have using (4.32)

$$\left|\frac{\partial v}{\partial x_i}\right| \leq r^{\gamma-1}\ C \tag{4.36}$$

and $v = 0$ for $r \geq \Delta$. Therefore

$$\int_{\hat{Q}} \left(\frac{\partial v}{\partial x_i}\right)^2\ dx \leq C \int_0^\Delta r^{2\gamma-2}\ r\ dr\ \leq C\ \Delta^{2\gamma} \tag{4.37}$$

We have also

$$\int_{\hat{Q}} |v|^2 dx \leq C \int r^{2\gamma+1} dr \leq\ C\ \Delta^{2\gamma+2} \tag{4.38}$$

Combining (4.37) and (4.38) we get (4.35).

From the point of view of applications the function $\rho(r)=r^{\gamma_0}\ g(|\ell gr|)$ is of importance especially with $g(x) = x^p$ or $g(x) = \cos x$ etc. Then (4.32) is satisfied with $\gamma = \gamma_0 - \varepsilon$, $\varepsilon > 0$ arbitrary.

4.4 The Convergence Rate of the p-Version of the Finite Element Method

Returning to our model problem (3.1) (3.2) we can assume (see e.g. [10], [15] that its solution u_0 can be written in the form

$$u_0 = \omega + \sum_{i=1}^{n} v_i \tag{4.39}$$

with $\omega \in H^k(\Omega_0) \cap H_0^1(\Omega_0)$ for the boundary condition $\Gamma u = u$ and $\omega \in H^k(\Omega_0)$ for the boundary condition $\Gamma u = \frac{\partial u}{\partial n}$ and

$$v_i = a_i \rho_i(r_i)\, \Theta_i(\phi_i) \in H_0^1(\Omega_0) \tag{4.40}$$

(resp. $H^1(\Omega_0)$) where r_i, ϕ_i are the polar coordinates with respect to the vertices A_i of the polygon Ω_0 and a_i are constants with

$$\rho_i(r) = r^{\gamma_i}\, g_i(|\ell g r_i|) \tag{4.41}$$

with

$$\left|\frac{\partial^j g_i(x)}{\partial x_j}\right| \leq C_i(j)\, x^{p_{i,j}} + D_i, \quad 0 < x < \infty \quad p_{i,j} \geq 0,\ j = 1,2,\ldots$$

and Θ_i is a function with all continuous and bounded derivatives. The coefficient γ_i is closely related to the angle in the boundary of Ω at the vertex A_i. Without any loss of a generality we can assume that Θ_i are smooth periodic functions with period 2π so that the function v_i is defined in the entire R^2. This form occurs in all elliptic problems e.g., elasticity, see [15].

Let now S be a triangularization of Ω_0 such that all vertices of Ω_0 are vertices of the triangulation. Obviously, we can assume that the support of $\rho_i(r)$ is arbitrarily small and v_i has support in an open cone K_j (with angle $< \pi$) so that the triangle T_j lies inside such a cone. See Fig. 4.2.

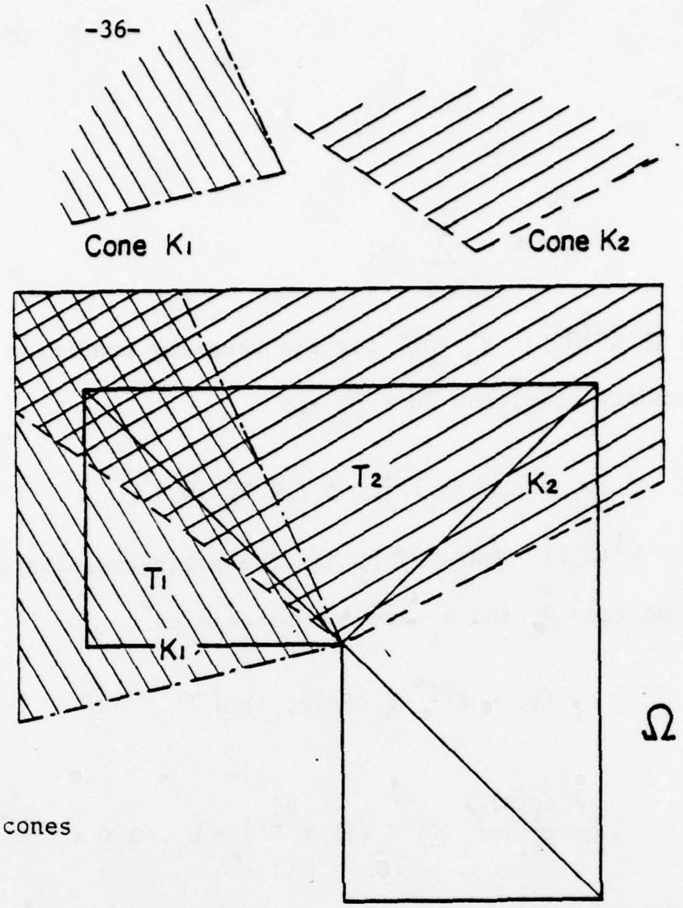


Figure 4.2
Triangles and enclosing cones

Denote now

$$V_\Delta = \sum_{i=1}^{n} v_i \, \chi_{i,\Delta} = \sum_{i=1}^{n} v_{i,\Delta}$$

where $\chi_{i,\Delta}$ is the function introduced in section 4.3 with respect to the origin (denoted by index i) of the polar coordinates of v_i. Now we are able to prove the major theorem of this section.

THEOREM 4.3. Let u_0 be the exact solution of the problem (3.1) and (3.2) which can be written in the form of (4.39), (4.40), (4.41) and let u_p be the finite element approximation. Then for $k > 1$

$$\| u_0 - u_p \|_{1,\Omega} \leq C(\varepsilon) \; p^{-\mu + \varepsilon} \, \|f\|_{k,\Omega}, \quad \varepsilon > 0 \text{ arbitrary} \tag{4.42}$$

$$\mu = \min\,[k-1,\; 2\gamma_i] \tag{4.43}$$

where $\gamma_i = \frac{\pi}{\alpha_i}$ and α_i is the angle of $\partial\Omega$ at the vertex A_i.

Proof. The exact solution u_0 can be written in the form (4.39) with $\omega \in H^{k+2-\varepsilon}(\Omega)$ and $\gamma_i = \frac{\pi}{\alpha_i}$. See [15]. It is sufficient to show that functions ω and v_i can be approximated by $z_p \in P_{p,0}^{[S]}$ resp. $z_p \in P_p^{[S]}$ (for boundary conditions $\Gamma u = u$ respectively $\Gamma u = \frac{\partial u}{\partial n}$) preserving the estimate (4.42).

Using theorems 3.2 and 3.1 we see that the function ω is approximable in the desired way and we have to concentrate only on approximation of the functions $v_i \in H_0^1(\Omega_0)$ resp. $H^1(\Omega_0)$. It is easy to see that $v_{i,\Delta} \in H_0^1(\Omega_0)$ if $v_i \in H_0^1(\Omega_0)$ and using lemma 4.5 we see that

$$\|v_{i,\Delta} - v_i\|_{1,\Omega_0} \leq C\Delta^{\gamma_i - \varepsilon} \tag{4.44}$$

In addition using theorem 4.2 and the remark to it, there exist polynomials

$$z_{p,j} \in P_p(T_j)$$

such that for any $k_i \geq 2\gamma_i + 1$

$$\| z_{p,j} - v_{i,\Delta}\|_{1,T_j} \leq C(\tilde{k}_i)p^{-(\tilde{k}_i-2)}\ \Delta^{-1/2\{\tilde{k}_i-2\gamma_i\}+1/2} \tag{4.45}$$

and $z_{p,j}$ satisfies also the condition (4.26) on the sides of T_j. The polynomials $z_{p,j}$ are not in general continuous through the sides of the triangles T_j, nevertheless because of condition (4.26) the function $z_{p,j_1} - z_{p,j_2}$ defined on the common sides of T_{j_1} and T_{j_2} satisfies the condition (4.26) too, it is a polynomial of degree p, and is zero at the end points of the side s. Using lemma 4.2 we can add a polynomial $\tilde{z}_{p,j}$ of degree p on T_{j_1} so that the continuity through the side s is accomplished and preserving the estimate

$$\| \tilde{z}_{p_i} - v_{i,\Delta}\|_{1,\Omega_0} \leq C(\tilde{k}_i)p^{-(\tilde{k}_i-2)}\ \Delta^{-1/2\{\tilde{k}_i-2\gamma_i\}+1/2} \tag{4.46}$$

In addition if $v_{i,\Delta} = 0$ on a side $s \subset \partial\Omega_0$, then $\tilde{z}_p = 0$ on this side also and so if $v_i \in H_0^1(\Omega_0)$ then $v_{i,\Delta} \in H_0^1(\Omega_0)$ and $z_{p_i} \in P_{p,0}^{[S]}(\Omega_0)$.

Combining now (4.44) and (4.46) we see that

$$\| z_{p_i} - v_i \|_{1,\Omega_0} \le C(\varepsilon)\ \Delta^{\gamma_i - \varepsilon} + C(\tilde{k}_i) p^{-(\tilde{k}_i - 2)}\ \Delta^{-1/2\{\tilde{k}_i - 2\gamma_i\}} + 1/2$$

Select now

$$\tilde{k}_i = k_0,\ k_0 \ge 2\gamma_i + 1$$

such that

$$\lambda = \frac{-2 + k_0}{(1/2)k_0 - (1/2)} \ge 2-\varepsilon$$

and

$$\Delta = p_i^{-\lambda} .$$

Then we have

$$\|z_{p_i} - v_i\|_{1,\Omega} \le C(\varepsilon) p_i^{-2\gamma_i + \varepsilon} + C(k_0) p^{-k_0 + 2 - \gamma_i \lambda + \lambda\{\frac{k_0}{2} - 1/2\}}$$

$$\le C(\varepsilon) p_i^{-2\gamma_i + \varepsilon'} + C(k_0) p^{-2\gamma_i + \varepsilon''}$$

$$\le C(\varepsilon) p^{-2\gamma_i + \varepsilon} \tag{4.47}$$

and so (4.47) yields immediately (4.42)(4.43).

So far we have analyzed the rate of convergence for the model problem (3.1) (3.2). It is obvious that the model problem was not an essential one. It was essential only to analyze the approximation behavior in $H^1(\Omega_0)$ resp. $H_0^1\ (\Omega_0)$.

Combining the main result of the theorem 4.3 and Theorem 3.6 we see that (up to ε) the estimate (4.42),(4.43) is a best possible.

5. NUMERICAL EXAMPLES

In order to illustrate the results of the theorems and in order to show the efficiency of p-version of the finite element, we now present several examples. The first example is a simple bar problem in one dimension and the numerical results are based on a computer program written specifically for this problem. The other examples are two dimensional and the numerical results are based on COMET-X, an experimental prototype for a general purpose finite element computer program developed at Washington University which implements the p-version of the finite element method [2].

5.1. A One Dimensional (Bar) Problem

We consider the problem: $' = \frac{d}{dx}$, $\Omega = (-1,1)$

$$u'' = -q(x) \quad \text{for } x \in \Omega \tag{5.1}$$

where the (loading) function q(x) and the (Dirichlet) boundary conditions will be specified later. The energy inner product is

$$B(u,v) = (u,v)_E = \int_{-1}^{1} u'(x)v'(x)dx. \tag{5.2}$$

We seek a solution $u \in H_0^1(\Omega)$ which satisfies

$$(u,v)_E = \int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} q(x)v(x)dx \quad \text{for all } v \in H_0^1(\Omega). \tag{5.3}$$

We choose as basis functions

$$\psi_i(x) = \int_{-1}^{x} P_i(t)dt \qquad i \geq 1$$

where $P_i(t)$ is the Legendre polynomial of degree i. Observe that $\psi_i(x)$, i=1,2,... form an orthogonal family with respect to the energy inner product i.e. $(\psi_i,\psi_j)_E = \int_{-1}^{1} P_i(x)P_j(x)dx = \frac{2}{2i+1}\delta_{ij}$.

In this one dimensional case it is possible to prove the direct and inverse approximation theorems by using the weighted Sobolev (respectively Besov) spaces associated with the Legendre differential equation:

$$-\frac{d}{dx}[(1-x^2)\frac{du}{dx}] = n(n+1)u$$

once we realize that the Legendre polynomials are eigenfunctions of the equation. Using this approach ε does not appear in the expressions for the rate of convergence, e.g. in (3.35) and (4.42). It is not clear how to generalize this idea to the two dimensional case. Our proof for two dimensions was therefore quite different.

First we consider convergence when Ω is not divided i.e. we use only one interval.

The finite element solution $u_p \in P^{[S]}_{p,0}(\Omega)$ satisfies

$$(u_p, \psi_i)_E = \int_{-1}^{1} q(x)\psi_i(x)dx \qquad i=1,2,\ldots p. \tag{5.4}$$

Writing

$$u_p(x) = \frac{1-x}{2}\, u(-1) + \frac{1+x}{2}\, u(1) + \sum_{i=1}^{p} a_i\psi_i(x)$$

it follows that

$$a_i = \frac{2i+1}{2}\int_{-1}^{1} q(x)\psi_i(x)dx \qquad i=1,2,\ldots p. \tag{5.5}$$

Also, denoting the error by

$$e_p(x) = u(x) - u_p(x) \tag{5.6}$$

it follows that in the energy norm $||e_p||_E^2 = (e_p, e_p)_E$

$$||e_p||_E^2 = ||u-u_p||_E^2 = ||u||_E^2 - ||u_p||_E^2 = ||\sum_{i=p+1} a_i\psi_i(x)||_E^2$$

$$= \sum_{i=p+1} a_i^2 \frac{2}{2i+1} \tag{5.7}$$

If we let $U = ||u||_E^2$ denote the strain energy then $U-U_p = ||e_p||_E^2$ is the error in strain energy.

<u>Case A</u> $\frac{du}{dx} = \sqrt{1-x^2}$, $q(x) = -\frac{d}{dx}\sqrt{1-x^2}$

In this case $u(x) = \frac{1}{2}(x\sqrt{1-x^2} + \sin^{-1}x)$ and the boundary conditions are $u(-1) = -\frac{\pi}{4}$, $u(1) = \frac{\pi}{4}$.

Also the energy is

$$||u||_E^2 = \int_{-1}^{1} (1-x^2)dx = \frac{4}{3} \ .$$

The coefficients a_i in (5.5) can be evaluated explicitly. First (5.5) becomes

$$a_i = \frac{2i+1}{2} \int_{-1}^{1} \sqrt{1-x^2}\ P_i(x)dx \tag{5.8}$$

Now $a_i = 0$ for i odd, and using the recurrence relation for derivatives of Legendre polynomials [1]

$$P'_{i+1}(x) - P'_{i+1}(x) = (2i+1)P_i(x)$$

for $i=2m$ $m=1,2,\ldots$ we obtain

$$\int_{-1}^{1}\sqrt{1-x^2}\ \ P_{2m}(x)dx = -\frac{1}{4m+1}\int_{-1}^{1}\frac{x}{\sqrt{1-x^2}}\ (P_{2m+1}(x) - P_{2m-1}(x))dx$$

$$= \frac{1}{4m+1}\int_0^{\pi}\cos\theta\ (P_{2m+1}(\cos\theta)-P_{2m-1}(\cos\theta))d\theta \tag{5.9}$$

From [1], page 785 formula (22.13.7) we have

$$\int_0^{\pi}(\cos\theta)P_{2m+1}(\cos\theta)d\ = \frac{\pi}{4^{2m+1}}\binom{2m}{m}\binom{2m+2}{m+1} \ . \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8) we obtain through straightforward caluclation

$$a_{2m} = \frac{2(2m)+1}{2} \frac{\pi}{4^{2m}2(m+1)(2m-1)} \binom{2m}{m}^2 .$$

Using Stirling's formula it follows that

$$\binom{2m}{m} \sim \frac{1}{\sqrt{\pi m}} 4^m ,$$

so that

$$a_{2m} = 0\left(\frac{1}{m^2}\right), \quad \text{as } m \to \infty. \tag{5.11}$$

Therefore, the square of energy of the error in (5.7) is given by

$$||e_p||_E^2 = \sum_{i=p+1} a_i^2 \frac{2}{2i+1} = 0\left(\sum_{i=p+1} \frac{1}{i^5}\right) = 0\left(\frac{1}{p^4}\right) = 0\left(\frac{1}{N^4}\right) \tag{5.12}$$

where N denotes the number of degrees freedom ($p \approx N$ is one dimension).

On the other hand, in order to study the convergence of the (usual) h-version with N linear uniformly distributed elements, let $x_i = -1 + \frac{2}{N}$ $i=0,1,2,\ldots N$ and let $u_h(x)$ denote the corresponding finite element solution. Then,

$$u_h(x_i) = u(x_i) \quad i=0,1,2,\ldots N$$

and we can compute the norm of the error $e_h = u(x) - u_h(x)$

We get (for linear elements)

$$0\left(\frac{1}{N^2}\right)|\lg N| \leq ||e_h||_E^2 = 0\left(\frac{1}{N^2}\right)|\lg N| \tag{5.13}$$

For quadratic and higher elements we get

$$0\left(\frac{1}{N^2}\right) \leq ||e_h|| = 0\left(\frac{1}{N^2}\right) \tag{5.14}$$

(In this respect see [3]).

Figure 5.1 shows in the log scale the behavior of the square of the energy error. We see that in the case of the p-version the rate is practically 4 as follows from the asymptotic analysis. In the case of the h-version the asymptotic range is not achieved and we see the rate about 1.81 instead 2

Case B $u(x) = |x|^{3/2}(1-x^2)$, $q(x) = -\frac{d^2}{dx^2}(|x|^{3/2}(1-x^2))$

The boundary conditions are $u(-1) = u(1) = 0$. The only qualitative difference between this case and case A is that the square root singularity in $u'(x)$ now occurs in the interior of Ω instead of at its boundary.

We again consider one interval using the same basis functions as before. (5.5) now becomes

$$\begin{aligned} a_i &= \frac{2i+1}{2}\int_{-1}^{1} \frac{d}{dx}(|x|^{3/2}(1-x^2))P_i(x)dx \\ &= \frac{2i+1}{2}\int_{-1}^{1} |x|^{1/2}\left(\frac{3}{2}-\frac{7}{2}x^2\right)(\text{sign } x)P_i(x)dx \\ &= \begin{cases} 0 & \text{if } i \text{ is even} \\ \frac{2i+1}{2}\int_0^1 x^{1/2}(3-7x^2)P_i(x)dx & \text{if } i \text{ is odd} \end{cases} \end{aligned} \tag{5.15}$$

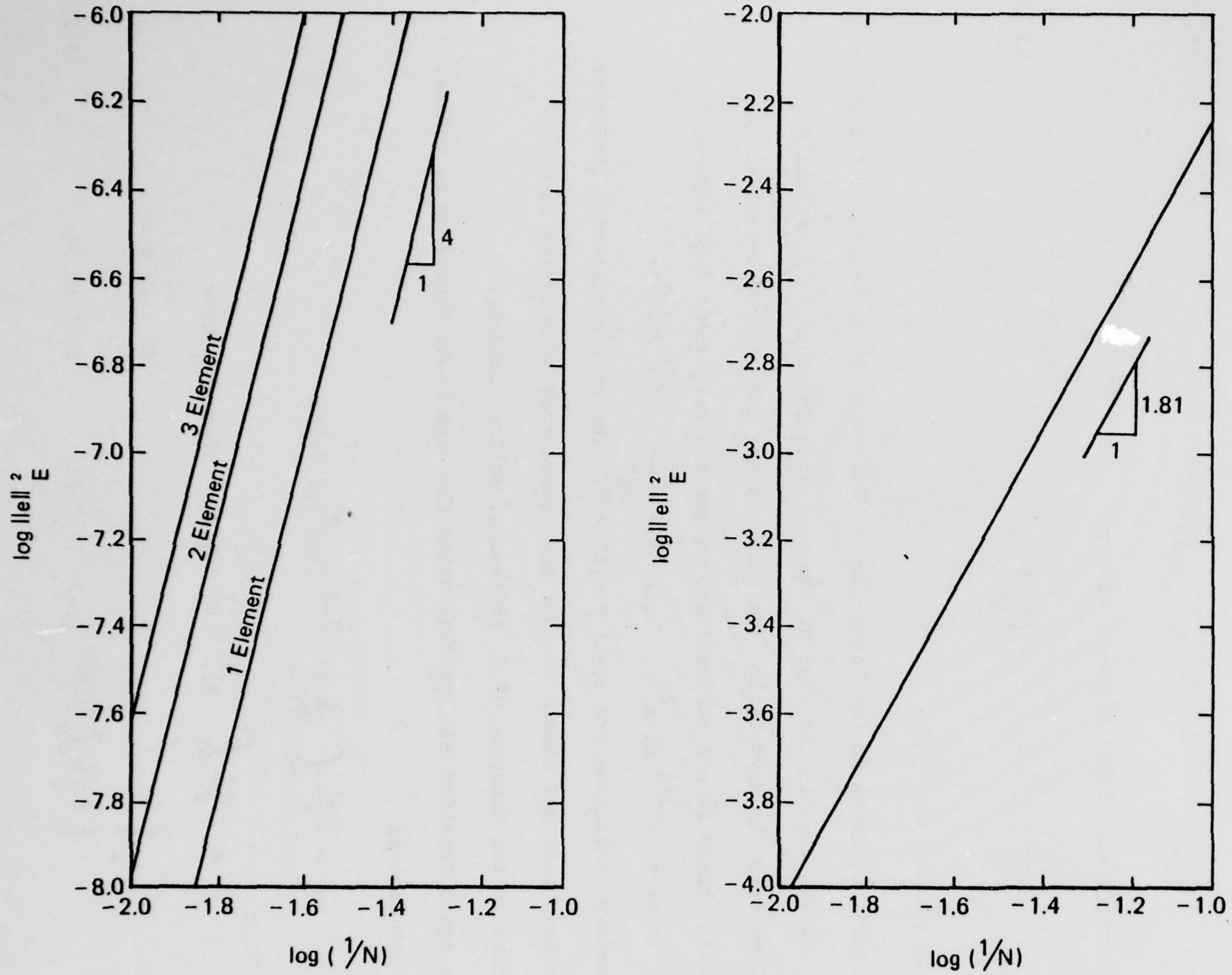


FIGURE 5.1

(a) p-version

(b) h-version

CASE A: Square of the energy error vs. reciprocal of the number of degrees of freedom

From [1], page 786, formula 22.13.9 we have

$$\int_0^1 x\, P_{2m+1}(x)dx = \frac{(-1)^m \Gamma(m + \frac{1}{2} - \frac{\lambda}{2})\Gamma(1 + \frac{\lambda}{2})}{2\Gamma(m + 2 + \frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})} \qquad \lambda > -2$$

so that after a straightforward calculation we obtain

$$\int_0^1 x^{1/2}(3-7x^2)P_{2m+1}(x)dx$$

$$= (-1)^m \left[\frac{\frac{3}{2}\Gamma(m + \frac{1}{4})\Gamma(\frac{5}{4})}{\Gamma(m + \frac{5}{4})\Gamma(\frac{1}{4})} - \frac{\frac{7}{2}\Gamma(m - \frac{3}{4})\Gamma(\frac{9}{4})}{\Gamma(m + \frac{13}{4})\Gamma(-\frac{3}{4})}\right]$$

$$= (-1)^{m+1}\left[\frac{\Gamma(m - \frac{3}{4})\Gamma(\frac{5}{4})(2m+1)(m+1)}{\Gamma(m + \frac{13}{4})\Gamma(-\frac{3}{4})}\right].$$

Substituting in (5.15) this yields

$$a_{2m+1} = (-1)^{m+1} \frac{\Gamma(m - \frac{3}{4})\Gamma(\frac{5}{4})(2m+1)^2(m+1)}{\Gamma(m + \frac{13}{4})\Gamma(-\frac{3}{4})}.$$

Using Stirling's formula it follows that for i odd

$$a_i = 0\left(\frac{1}{i}\right) \qquad \text{as } i \to \infty.$$

Therefore, the square of the energy of the error is now given by

$$||e_p||_E^2 = \sum_{i=p+1} a_i^2 \frac{2}{2i+1} = 0\left(\sum_{i=p+1} \frac{1}{i^3}\right) = 0\left(\frac{1}{p^2}\right) = 0\left(\frac{1}{N^2}\right)$$

which has the same rate of convergence (up to log term) as the square of the error $||e_h||_E^2$ for the h-version. This illustrates the importance of the statement made at the end of section 4 that in order to get the full power of the p-version, singularities must be located at vertices of the finite element mesh.

To illustrate this point further, we plot in figures 5.2, $||e_h||_E^2$ and $||e_p||_E^2$ for case B, using one, two and three equal intervals for the p-version of the finite element solution. The results are summarized in Table 5.1. The convergence of the h-version remains the same ($||e_h||_E^2 = 0(\frac{1}{N^2} \lg N)$ for both cases A and B. In case A the convergence of the p-version remains the same regardless of the number of intervals ($||e||_E = 0(\frac{1}{N^2})$) whereas in Case B, the order is 2 for <u>two</u> intervals, whereas it is only 1 for both one and <u>three</u> intervals. This is, of course, because in case B for two intervals the singularity is at a vertex of the mesh whereas for one and three intervals it is in the interior of elements of the mesh, with the consequent degrading of rate of convergence. In case A the singularity is always at a vertex of a mesh. We mentioned here only the case of the h-version with <u>uniform</u> mesh spacing. It can

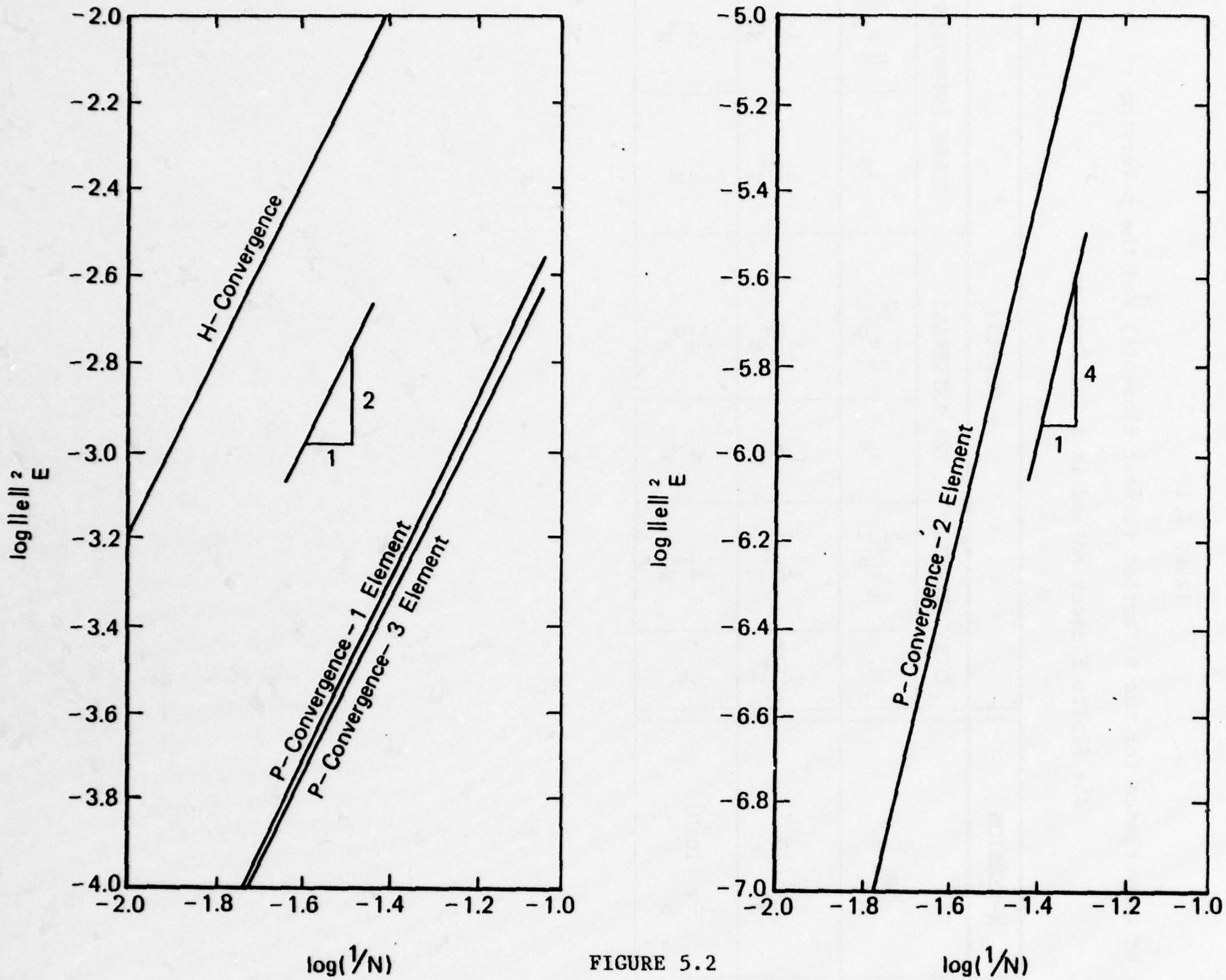


FIGURE 5.2

(a) Singularity in the interior of the element (b) Singularity at element boundary

CASE B: Effect of the location of singularity on the rate of convergence

TABLE 5.1

Rates of Convergence for the h-Version (Linear Elements) and the p-Version of the Finite Element Method in a Bar Problem

	h-VERSION	p-VERSION					
		ONE INTERVAL		TWO INTERVALS		THREE INTERVALS	
	$\|e_h\|_E^2$	a_N	$\|e_p\|_E^2$	a_N	$\|e_p\|_E^2$	a_N	$\|e_p\|_E^2$
CASE A	$\frac{1}{N^2}\,\|\ln N\|$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$
CASE B	$\frac{1}{N^2}\,\|\ln N\|$	$\frac{1}{N}$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N}$	$\frac{1}{N^2}$

that when an optimal nonuniform mesh spacing for elements of degree p(fixed) is used then $||e_h||_E^2 = 0\left(\frac{1}{N^p}\right)$ where function 0 is independent of N, but depends on p. In this very special case, it is possible, of course, to analyze in more detail the combined h-p-version, but we shall not go into that.

5.2. Two Dimensional Problems - An Edge Cracked Panel and a Parabolically Loaded Panel

We now consider two problems taken from two-dimensional linear elasticity. One is an edge cracked rectangular panel, shown in Figure 5.3, the other is the parabolically loaded square panel, shown in Figure 5.4. In both bases the displacement field is of the form $\underline{u} = r^{\alpha}\underline{\phi}(\theta)$, where r and θ are polar coordinates and $\underline{\phi}$ is a smooth function. In the case of the edge cracked panel $\alpha = \frac{1}{2}$ when r is measured from the crack tip; in the case of the parabolically loaded panel $\alpha \cong 2.74$ when r is measured from the corner of the panel (See [27].). The computations were performed with the computer program COMET-X which allows the polynomial order p to be varied between 1 and 8. We wish to illustrate the following points:

(a) As claimed by the theoretical results, the rate of convergence is

$$U - U_p \sim CN^{-2\alpha} \tag{5.16}$$

(when neglecting ε and the fact that the edge cracked panel is not a Lipschitzian domain). In Figure 5.5 we plotted $\tilde{U} - U_p$ vs N^{-1} on log - log scale for the edge cracked panel for two x/a ratios. $\tilde{U}$ is an estimate of the exact strain energy value (of the half panel) obtained by extrapolation from the following

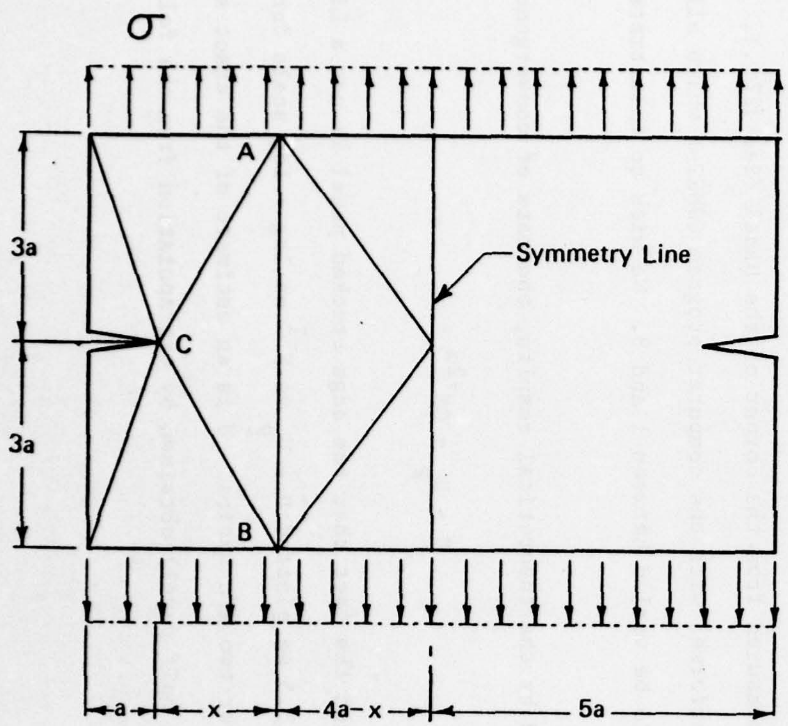


FIGURE 5.3

Edge cracked rectangular panel and finite element triangulation

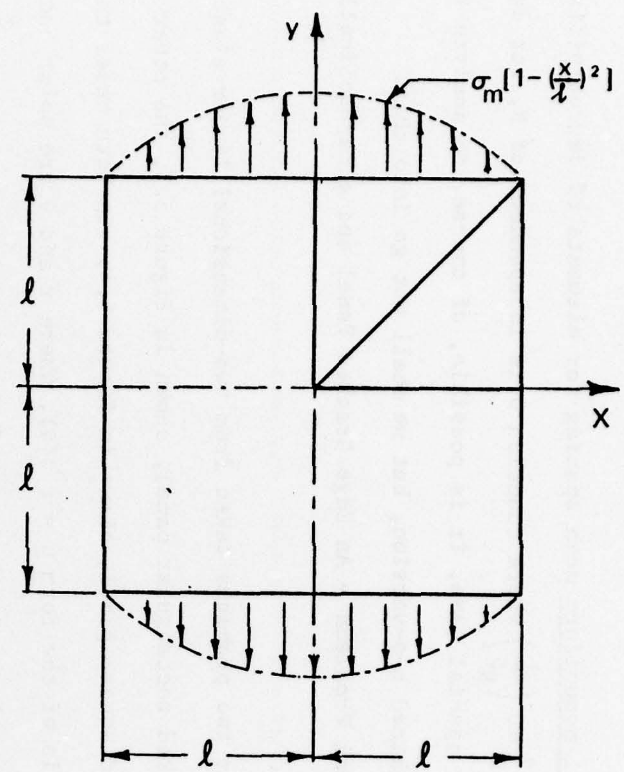


FIGURE 5.4

Parabolically loaded square panel and finite element triangulation

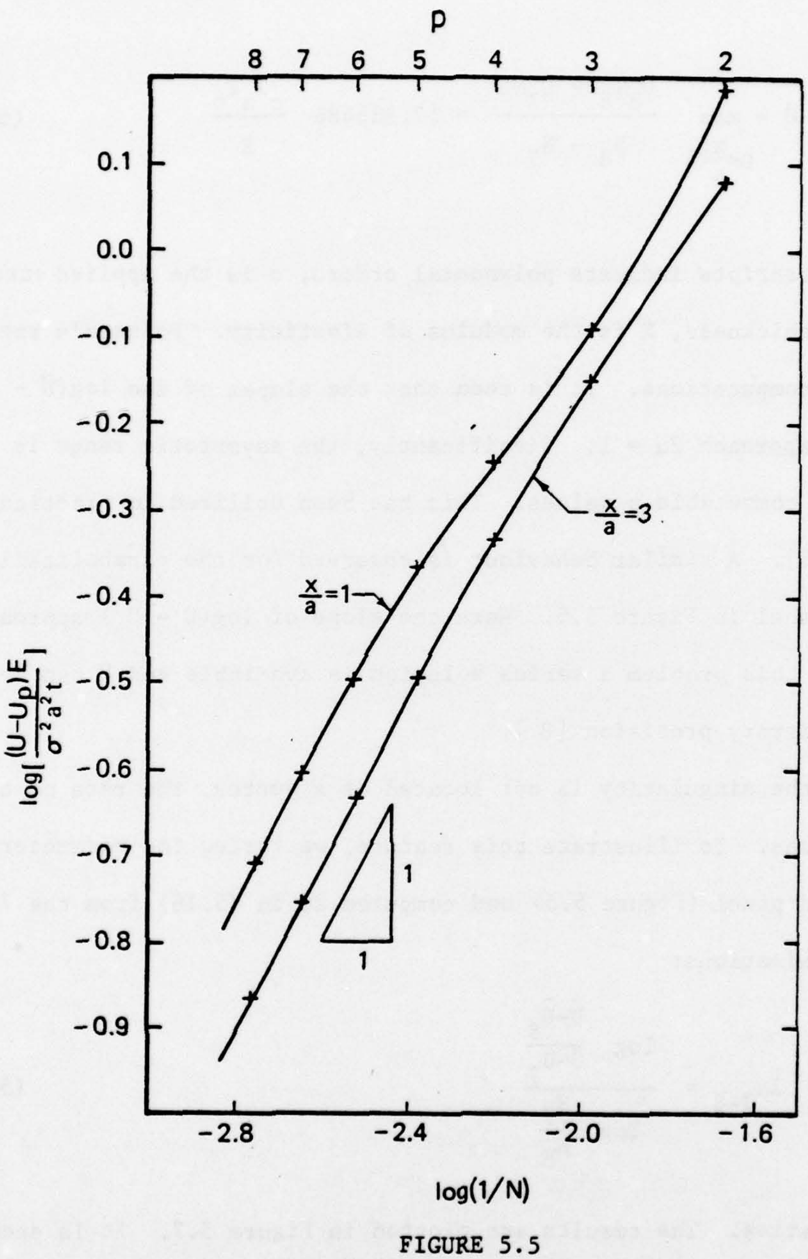


FIGURE 5.5

Edge cracked rectangular panel. Estimated error in strain energy vs. reciprocal of the number of degrees of freedom

expression:

$$\tilde{U} = \max_{0<\frac{x}{a}<4} \frac{U_8 N_8 - U_7 N_7}{N_8 - N_7} = 17.385486 \frac{\sigma^2 a^2 t}{E} \tag{5.17}$$

in which the subscripts indicate polynomial orders, σ is the applied stress, t is the panel thickness, E is the modulus of elasticity. Poisson's ratio was 0.3 in all computations. It is seen that the slopes of the $\log(\tilde{U} - U_p)$ curves rapidly approach $2\alpha = 1$. Significantly, the asymptotic range is entered at low, computable p values. This has been utilized in practical computations [26]. A similar behaviour is observed for the parabolically loaded square panel in Figure 5.6. Here the slope of $\log(U - U_p)$ approaches $2\alpha = 5.48$. For this problem a series solution is available and U can be computed to arbitrary precision [8].

(b) When the singularity is not located at a vertex, the rate of convergence decreases. To illustrate this feature, we varied the parameter x for the edge cracked panel (Figure 5.5) and computed 2α in (5.16) from the 7th and 8th order approximations:

$$2\alpha_{7-8} = \frac{\log \dfrac{\tilde{U}-\tilde{U}_8}{\tilde{U}-U_7}}{\log \dfrac{N_7}{N_8}} \tag{5.18}$$

for various $\frac{x}{a}$ ratios. The results are plotted in Figure 5.7. It is seen that $2\alpha_{7-8}$ decreases as the interelement boundary approaches the crack tip C. It was found that aspect ratios as high as 300 could be employed without encountering numerical instability.

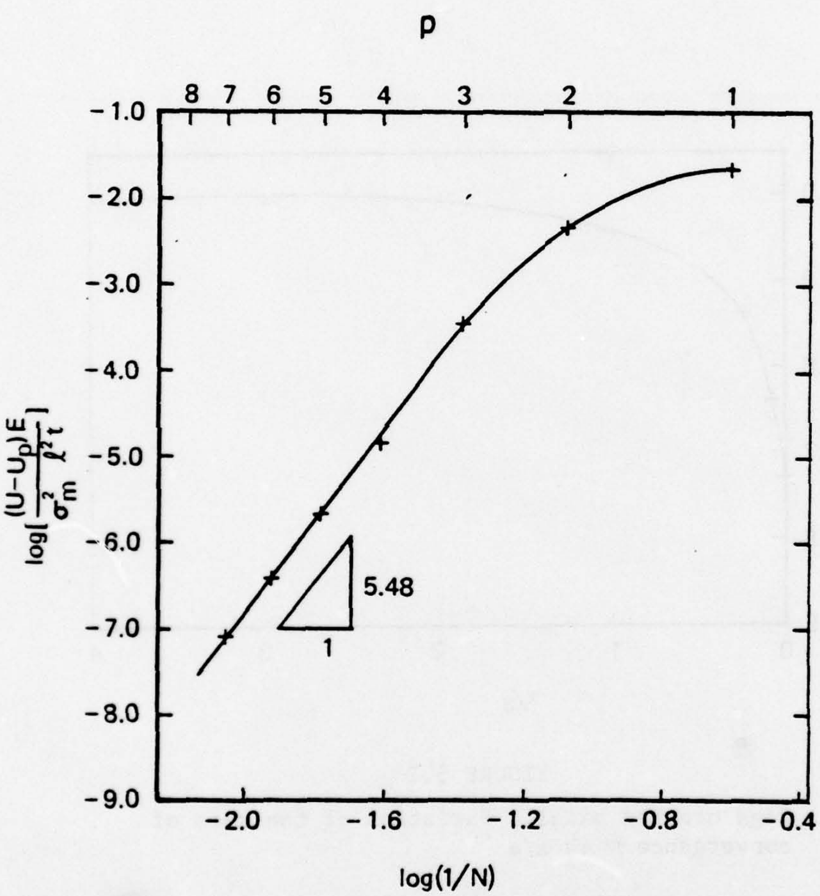


FIGURE 5.6

Parabolically loaded square panel. Error in strain energy vs. reciprocal of the number of degrees of freedom

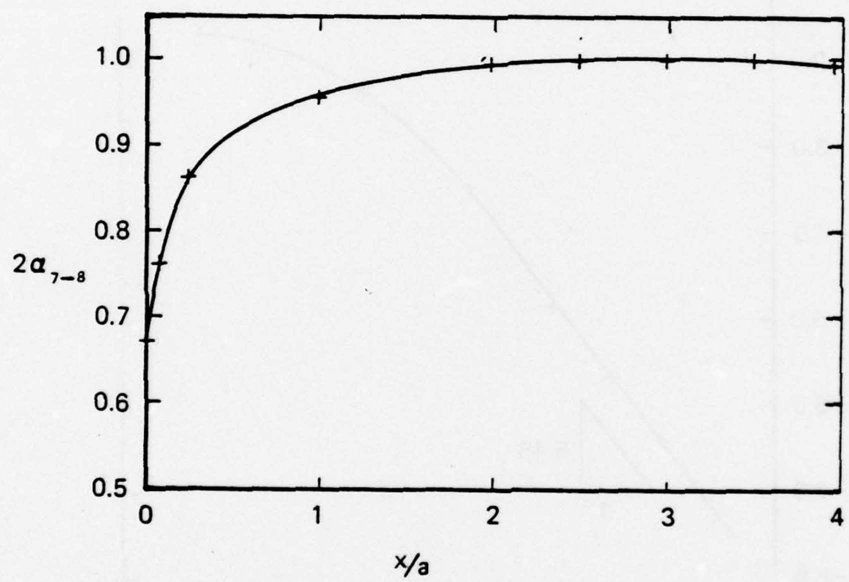


FIGURE 5.7

Edge cracked panel. Variation of the rate of convergence with x/a

5.3. Round-off error

When high order polynomials are used, the choice of basis functions becomes important from the point of view of round-off error. It is possible to design stable basis functions on the basis of theory developed mainly by Mikhlin - see [16, Chapter 2] and [4, Chapter 4,7]. Of course, the choice of basis functions is also influenced by programming considerations and the range of p for which the program is written. In general, it is desirable that the basis functions be hierarchic, as described in Section 6.1., and computation of elemental stiffness matrices and load vectors be as simple as possible.

The basis functions currently in COMET-X were chosen primarily on the basis of programming considerations and they are not optimal from the point of view of round-off error. Experience with the code has not indicated significant accumulation of round-off error, however, in double precision computations within the range of p allowed by COMET-X (1 to 8).

To study the characteristics of these basis functions, from the point of view of round-off error, the assembly and elimination procedures were executed in both double and single precision (7 resp. 15 decimals on the DEC System 20 computer) for the two problems described in Section 5.2. All other computations were performed in double precision only. (COMET-X employs a modified version of Irons' frontal solver [11] to carry out assembly and elimination). The results, given in Table 5.2 indicate that for $p \leq 8$ the round-off error is not critical but if significantly higher p is to be used then it will be necessary to exercise caution in selecting the basis functions.

TABLE 5.2

Accumulation of Round-Off Error with Increasing Polynomial Order

p	EDGE CRACKED RECTANGULAR PANEL (FIG. 5.3)						PARABOLICALLY LOADED SQUARE PANEL (FIG. 5.4)		
	x/a = 3			x/a = 0.2					
	U_{DP}	U_{SP}	log ε	U_{DP}	U_{SP}	log ε	U_{DP}	U_{SP}	log ε
5	17.16690	17.16690	< -6	16.83520	16.83520	< -6	0.2542124	0.2542124	< -6
6	17.25347	17.25329	- 4.99	16.97998	16.98000	- 5.94	0.2542144	0.2542144	< -6
7	17.31114	17.31670	- 3.50	17.04281	17.07217	- 2.78	0.2542147	0.2542147	< -6
8	17.34988	17.38613	- 2.68	17.12741	17.08960	- 2.67	0.2542148	0.2542170	-5.06

U_{DP}: computed strain energy, all computations in double precision;

U_{SP}: computed strain energy, assembly and elimination in single precision;

$$\varepsilon = \frac{|U_{DP} - U_{SP}|}{U}$$

U: exact (or estimated true) strain energy

$U = 17.485486 \dfrac{\sigma^2 a^2 t}{E}$ for the edge cracked panel

$U = 0.25421481 \dfrac{\sigma_m^2 \ell^2 t}{E}$ for the parabolically loaded square panel

6. COMPUTER IMPLEMENTATION OF THE P-VERSION: COMET-X

In order to implement the p-version efficiently it is necessary to have available a family of finite elements of arbitrary polynomial degree having certain properties. The family should allow, for example, as much information to be carried over as possible when increasing the degree from p to p+1. The present version of COMET-X contains a family of triangular finite elements which enforce C° continuity across interelement boundaries for problems which require solutions in $H_0^1(\Omega)$ (planar elasticity). We now describe some of the salient features of COMET-X.

6.1 Hierarchic Property of Basis Functions

The basis functions corresponding to an approximation of degree p constitute a subset of those corresponding to an approximation of degree p+1. Therefore, the stiffness matrix of the element of degree p is embedded in the stiffness matrix of the element of degree p+1. All calculations performed in generating the p<u>th</u> order elemental stiffness matrices and load vectors can be saved for use in the (p+1)st degree calculation. We call this the hierarchic property of the family.

As an illustration of the difference between conventional and hierarchic basis functions, consider linear and quadratic C° basis functions for a triangle (given in natural coordinates (L_1, L_2, L_3); see [20] for a discussion of natural coordinates). The linear function which is one at vertex i and zero at the other two vertices is L_i i=1,2,3 and it is the basis function for the nodal variable u(i) i = 1,2,3. In defining quadratic approximations,

conventional approaches use the nodal variables $u(i)$, $u(i')$ $i,i' = 1,2,3$ where i' is the midpoint of side i (opposite vertex i). It is clear that the basis functions corresponding to $u(i)$ $i = 1,2,3$ change from the linear to the quadratic approximation. In the hierarchic approach, the nodal variables used for the quadratic approximation are $u(i)$, $u_{ss}(i')$ where the subscript s denotes the differentation in the direction of a side. For $p \geq 3$, the external nodal variables used to enforce C° continuity are jth order derivatives at the midpoint of each side in the direction of the side $3 \leq j \leq p$. Other nodal variables (called internal nodal variables) are used to complete the polynomial to one of degree p. See [12 ,13,14,19,20,21,22].

6.2 Precomputed Arrays

It is possible to compute certain elemental stiffness submatrices (corresponding to a standard triangle [14]) once and for all, and then to use these standard submatrices in order to calculate the element stiffness matrices in a given problem. Precomputed arrays based on hierarchic families permit convenient use of elements of different polynomial degrees in the same mesh because two elements of different degree are easily matched along an interelement boundary. The precomputed standard submatrices are also hierarchic in character so that one version of these arrays, corresponding to the maximum polynomial degree that will be used, can be easily stored on a permanent file. Precomputed arrays are described in [23] and have been incorporated into COMET-X.

6.3 Computational Cost

There are three main phases in the computational process of the finite element method:

a) Input phase; which includes the computations of elemental stiffness matrices and load vectors;

b) Solution Phase; which comprises the assembly and elimination processes;

c) Output phase, which includes the computation of displacements, stresses, etc.

When the number of degrees of freedom is progressively increased, the major variable cost occurs in the solution phase. In a number of numerical experiments performed with COMET-X it was found that the CPU time for the solution phase can be closely approximated by an expression of the form $a + bN^{\beta}$; $2 < \beta < 2.4$, a and b constants. Thus, although the stiffness matrix tends to be more fully populated in the p-version than in the h-version, sparse matrix solution techniques have provided substantial reduction in the number of operations as compared with solvers which do not account for sparsity ($\beta=3$). As has been already noted, the solution technique in COMET-X is similar to Irons' frontal solver technique.

Solution time information is given in Fig. 6.1 for the edge cracked rectangular panel ($x/a = 3$). The computations were performed in double precision on a DEC-20 computer, (DEC System 2040, 128K 36 bit word memory, TOPS-20 operating system). The time for the frontal solver includes both the assembly and elimination procedures. The time is given in both CPU seconds and in Equivalent Time Units. (ETU). As in [23], an ETU is the time required for squaring a full 18 x 18 matrix by means of the subroutine GMPRD (double precision) of the IBM Subroutine Package. On the DEC-20 computer this operation requires approximately 0.33 seconds.

The total time accounts for all three phases of the computation, including computation of the displacement vector and stress tensor at six points per element.

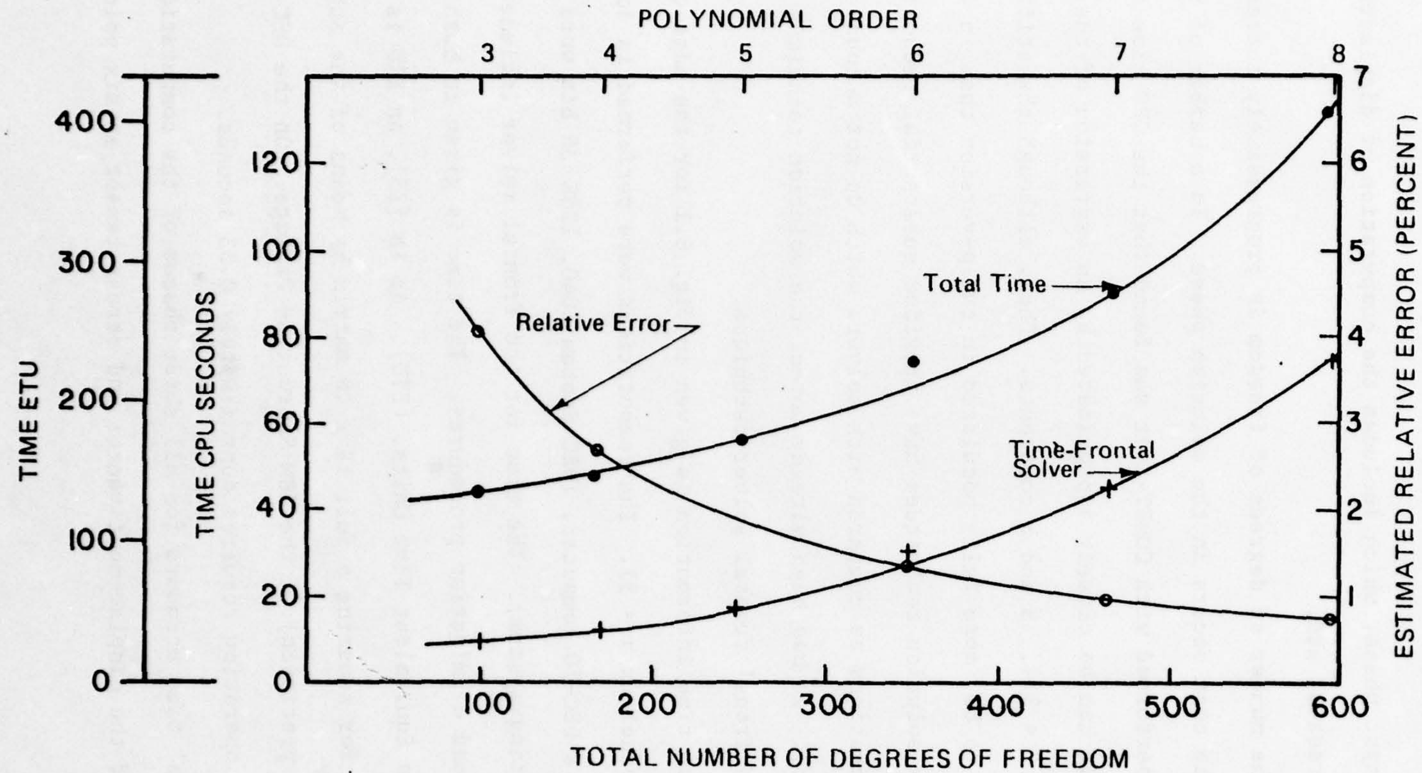


FIGURE 6.1

Edge cracked panel, $\frac{x}{a} = 3$. Variation of the estimated relative error in strain energy and solution time with the number of degrees of freedom. The relative error is defined as $\frac{\tilde{U}-U_p}{\tilde{U}} \times 100$ where $\tilde{U}$ is given by eq. (5.17) and U_p is the computed strain energy.

6.4 The h and p versions of the finite element method

Let us compare the h and p versions of the finite element method on the basis of the present state of theory and experience.

1) Asymptotic rate of convergence (in energy) with respect to the number of degrees of freedom:

   a) <u>For smooth solutions</u> the rate of convergence of the p-version is not limited by fixed polynomial degree, as in the h-version.

   b) In the case of <u>nonsmooth solutions</u>, the p-version has at least the same rate of convergence as the h-version (when the h-version is based on quasiuniform mesh refinement) but in practical cases, for example when the singularity is caused by corners, the rate of convergence of the p-version is twice that of the h-version.

   c) The h-version, coupled with optimal mesh design, results in higher convergence rate; however, the p-version can also be used in conjunction with optimally designed meshes. In this regard, the mesh design seems to be much less critical for the p-version than for the h-version.

2) Input: Because relatively few elements are used in the p-version, the volume of input data is smaller for the p-version than for the h-version.

3) Round-off: In practical cases the round-off problem does not appear to be more critical for the p-version than for the h-version.

4) Flexibility: From the practical, rather than the theoretical point of view, the flexibility of the p-version is somewhat restricted by the fact that constant coefficients are assumed over large finite element domains. At the present there is insufficient experience with curvilinear and other numerically integrated elements in connection with the p-version.

5) Solution time: The available experience indicates that for a given number of degrees of freedom the solution time for the p-version is about the same as for the h-version.

6) Adaptivity: Development of adaptive finite element procedures has now been recognized as an important area for research. (See, for example, [18]). From the point of view of implementation, adaptivity based on the p-version appears to be simpler. Adaptivity based on the h-version poses difficult data management problems. See, for example [5 ,28]. In principle, it is possible to base adaptivity on a combination of the h- and p-versions but such an approach would again pose difficult data management problems. A more promising approach is to employ mesh grading on a prior basis, either manually or with standard mesh generators, and then to make adaptive changes by means of adjusting p.

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HIERARCHIC FAMILIES FOR THE P-VERSION OF THE FINITE ELEMENT METHOD

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Summary

The p-version of the finite element method is a new approach to finite element analysis in which the partition of the domain is held fixed while the degree p of approximating piecewise polynomials is increased. In this paper, two theorems are presented which describe the approximation properties of the p-version. In particular, for the singularity problem, the p-version has asymptotically as $p \to \infty$ twice the order of convergence of the standard version of the finite element method, if the number of degrees of freedom is used as a measure of convergence. Various hierarhic families of finite elements, designed for computationally efficient computer implementation of the p-version, are described. These families include conforming C^0 and C^1 triangular families, and conforming C^0 families of rectangles and tetrahedra.

Introduction

In the finite element method the solution to a certain type of partial differential equation is formulated as a variational problem. In the conventional approach the solution is then approximated over the given domain by functions which are piecewise polynomials on convex subdomains (such as triangles or rectangles) and which are globally in C^n, $n \geq 0$, where n depends upon the order of the partial differential equation. The degrees of the approximating piecewise polynomials are fixed (usually at some low number such as 2 or 3) and the accuracy of the approximation is increased by allowing h, the maximum diameter of the finite elements, to go to zero. We refer to this approach as the h-version of the finite element method. The h-version has been studied extensively and asymptotic error bounds as $h \to 0$ are well known for its rate of convergence.

In a new approach developed at the Center of Computational Mechanics at Washington University a different point of view is adopted. The given domain is partitioned into convex subdomains which are kept fixed, and the solution is again approximated by functions which are globally in C^n, $n \geq 0$, and which are polynomials over each convex subdomain, Now, however, accuracy is increased by allowing the degree p of the piecewise polynomials to go to infinity. We call this approach the p-version of the finite element method. The p-version is reminiscent of the classical Ritz method but with one important difference. In the Ritz method the solution is approximated on the entire domain by polynomials (or other smooth functions) whereas in the p-version of the finite element method it is approximated only over each convex subdomain by a polynomial and globally the approximate is required to be in C^n. This difference leads to a rate of convergence for the p-version which is higher than that of both the Ritz method and the h-version, and also to other computational advantages.

1.1. A Sample Problem

In order to illustrate the application of the p-version of the finite element method to a practical situation, we consider a sample problem, called Lockheed Test Problem No. 2, which has been used as a test case for various finite element programs.[1] It consists of a circular cylindrical shell with symmetrically loaded cutouts and subjected to a uniform axial end shortening of known amount (Figure 1).

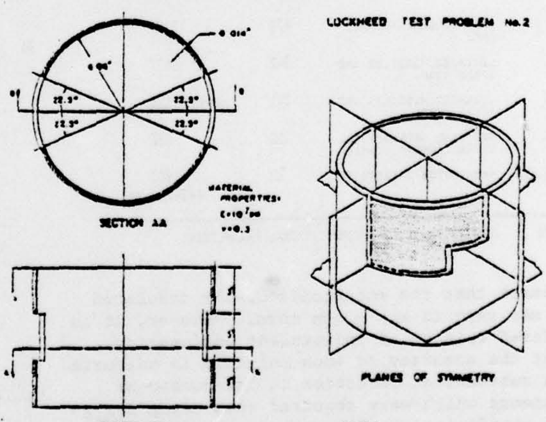

FIGURE 1

The shell is made of a homogeneous isotropic linearly elastic material and has constant thickness. u, v, and w represent the longitudinal, circumferential, and normal displacements, respectively. The boundary conditions at the ends of the shell are

$$w = v = \frac{\partial w}{\partial x} = 0 \qquad u = \text{constant} = 0.2 \times 10^{-3} \text{ inches}$$

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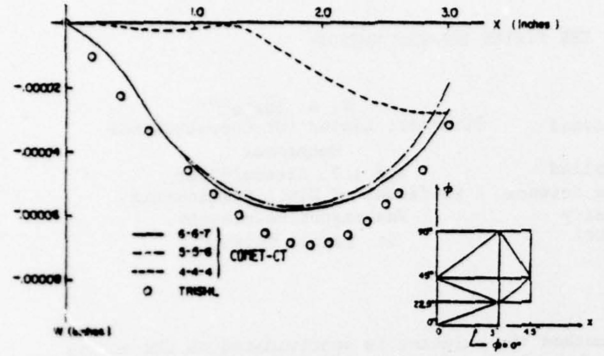


FIGURE 2: NORMAL DISPLACEMENT ALONG CENTER LINE OF SHELL (φ=0)

The 10-element triangulation used to solve this problem is shown in Figure 2, together with the normal displacement along the center line of the shell. The numbers 4-4-4, 5-5-6, 6-6-7 refer to the degrees of the polynomials used to approximate u,v, and w. In this problem the approximations to u and v are globally C^0, and the approximation to w is globally C^1. In Figure 3 a comparison is given of the numbers of elements and degrees of freedom used in the application of several computer programs to this problem. The name COMET-CT refers to an early experimental implementation of the p-version of the finite element method.[2]

COMPUTER CODE	SOURCE	NUMBER OF ELEMENTS	DEGREES OF FREEDOM
SHELL 3	GULF GENERAL ATOMIC, INC.	476	2457
STAGS	LOCKHEED MISSILES AND SPACE CORP.	342	1437
REXBAT	LOCKHEED MISSILES AND SPACE CORP.	241	APPROX 1125
TRISHL	NATIONAL AERONAUTICAL ESTABLISHMENT, CANADA	100	637
COMET-CT	WASHINGTON UNIVERSITY	10	500 (6-6-7-CASE)

FIGURE 3: COMPARISON OF SOME COMPUTATIONAL PARAMETERS

We remark that the accuracies of the tabulated solutions may vary in any given norm. However, it is the considered opinion of independent engineering groups that the accuracy of each solution is adequate. There is a substantial reduction in the number of finite elements which were required when using the p-version of the finite element method, to solve this problem. Additional details on the solution of this problem are given by Rossow et al.[2]

In this paper we consider the uniform (or quasi-uniform) h-version and the uniform p-version of the finite element method i.e. the partition of the domain is refined uniformly (or quasi-uniformly) while the degree p of approximating piecewise polynomials is held fixed (in the h-version), whereas the partition is held fixed while the degree p is increased (in the p-version). A general theory for combining both the h- and p-versions in a simple manner is not yet developed.

Early results on the convergence of the p-version of the finite element were empirical based largely on numerical experiments [2,3,4,5,6,7]. Recently [8], a firm mathematical foundation was provided for the p-version, in which basic approximation properties were derived. We first state two theorems proved in [8] which establish the rate of convergence of the p-version of the finite element method. Then, we describe several new families of finite elements designed to implement the p-version on computers. These families have a hierarchic property which leads to computational savings when using the p-version.

2. Theoretical Background and Illustrations

2.1. Theoretical Background

Let Ω be a bounded polygonal domain in two dimensional Euclidean space R^2, let $E(\Omega)$ be the space of all real C^∞ functions on Ω with continuous derivatives of all orders on $\bar{\Omega}$, let $D(\Omega) \subset E(\bar{\Omega})$ be the subspace of functions with compact support in Ω, let $H^0(\Omega)=L_2(\Omega)$ with the inner product

$$(u,v)_{0,\Omega} = \int_\Omega uv dx, \quad dx = dx_1 dx_2.$$

For $k \geq 1$ integral let $H^k(\Omega)$ resp. $H_0^k(\Omega)$ be the completions of $E(\bar{\Omega})$ resp. $D(\Omega)$ under the norm

$$\|u\|^2_{k,\Omega} = \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{0,\Omega}$$

where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$, $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \geq 0$ integral, $i = 1,2$, $|\alpha| = \alpha_1 + \alpha_2$. The inner product in $H^k(\Omega)$ will be denoted by $(\cdot,\cdot)_{k,\Omega}$. For $k > 0$ nonintegral the space $H^k(\Omega)$ and $H_0^k(\Omega)$ are defined by interpolation.

Consider the following model problem

$$-\Delta u + u = f \quad \text{on } \Omega, \quad f \in H^0(\Omega) \tag{2.1}$$

$$\Gamma u = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

where Ω is a bounded polygonal domain and $\Gamma u = u$ or $\Gamma u = \partial u/\partial n$. We seek a solution in the weak sense i.e. $u_0 \in H_0(\Omega)$ such that

$$B(u_0,v) = (f,v) \quad \text{for all } v \in H_0^1(\Omega)$$

$$(\text{resp. } u \in H^1(\Omega)) \tag{2.3}$$

where we define

$$B(u_0,v) = (u_0,v)_{1,\Omega} \tag{2.4}$$

The concept of convergence in the p-version of the finite element method is now formulated as follows: Let S be a (fixed) triangulation of Ω, $S = \{T_i\}$ $i = 1,\ldots,m$ where T_i are open triangles such that $\cup \bar{T}_i = \bar{\Omega}$ and T_i, T_j $i\neq j$ have either a common entire side or a vertex or $T_i \cap T_j = \emptyset$. Denote by $P_p^{[S]}(\Omega) \subset H^1(\Omega)$ the subset of all functions $u \in H^1(\Omega)$ such that if $u_{(T_i)}$ is the restriction of u to T_i then $u_{(T_i)} \in P_p(T_i)$ i.e. $P_p^{[S]}(\Omega)$ consists of all functions which are piecewise polynomials of degree at most p and which belong to $H^1(\Omega)$. Further, let $P_{p,0}^{[S]}(\Omega) = P_p^{[S]}(\Omega) \cap H_0^1(\Omega)$. The p-version of the finite element method consists of finding u_p $p = 1,2,\ldots$ where $u_p \in P_{p,0}^{[S]}(\Omega)$ (resp. $P_p^{[S]}(\Omega)$) (for the boundary conditions $u = 0$ resp. $\Gamma u = (\partial u/\partial n)$ so that (2.3) holds for all $v \in P_{p,0}^{[S]}(\Omega)$ (resp. $P_p^{[S]}(\Omega)$).

A bound for the rate of convergence in the p-version of the finite element is given by the following. theorem:

<u>Theorem:</u> Let $u \in H^k(\Omega)$, $k > 1$, be the exact solution of the problem (2.1), (2.2) and let u_p be the finite element approximation, then

$$\|u_0 - u_p\|_{1,\Omega} \leq C(k,\varepsilon)p^{-(k-1)+\varepsilon}\|u_0\|_{k,\Omega} \quad (2.5)$$

where $\varepsilon > 0$ is arbitrary. For the boundary condition $\Gamma u = (\partial u/\partial n)$, ε can be set to zero.

Now, a polynomial of degree p has N degrees of freedom with $N \approx p^2$. Therefore, $P_p[S]$ and $(P_p[S]_0)$ has dimension N with $N \approx p^2$ and (2.5) can be rewritten in the form

$$\|u_0 - u_p\|_{1,\Omega} \leq C(k,\varepsilon)N^{-\frac{(k-1)}{2}+\varepsilon}\|u_0\|_{k,\Omega_0} \quad (2.6)$$

On the other hand, for the h-version of the finite element method with quasi-uniform mesh we have

$$\|u_0 - u_h\|_{1,\Omega} \leq Ch^{\mu}\|u_0\|_{k,\Omega}, \quad \mu = \min(k-1,q) \quad (2.7)$$

where q is the degree of the approximating polynomial used in the elements. Recalling that in this case, $N \approx h^{-2}$ we can rewrite (2.7) in the form

$$\|u_0 - u_h\|_{1,\Omega} \leq C\, N^{-\frac{\mu}{2}}\|u_0\|_{k,\Omega}. \quad (2.8)$$

This rate of convergence is optimal (possibly up to $\varepsilon > 0$). Comparing (2.7) and (2.8) we see that the p-version gives results which are (neglecting $\varepsilon > 0$) not worse than the h-version with quasi-uniform mesh if we compare the number of degrees of freedom leading to the same rate of convergence. Also the convergence rate can be much better because in the p-version we do not have the restriction due to the degree of the elements as we have in the h-version. Further, in many practical situations, the factor 2 in (2.6) can be removed, and then the p-version will be superior to the h-version with quasi-uniform mesh.

More specifically we consider the p-version when used to solve a singularity problem. Assume that the solution u_0 to our model problem (2.1), (2.2) can be written in the form

$$u_0 = \omega + \sum_{i=1}^{m} v_i \quad (2.9)$$

with $\omega \in H^k(\Omega) \cap H_0^1(\Omega)$ for the boundary condition $\Gamma u = u$ and $\omega \in H^k(\Omega)$ for the boundary condition $\Gamma u = (\partial u/\partial n)$,

$$v_i = a_i \rho_i(r_i)\theta_i(\phi_i) \subset H_0^1(\Omega) \quad (\text{resp. } H^1(\Omega)) \quad (2.10)$$

where r_i, ϕ_i are polar coordinates with respect to the vertices of the polygon Ω and a_i are constants, and where

$$\rho_i(r) = r_i^{\gamma_i} g_i(|\log r_i|) \quad (2.11)$$

with

$$\left|\frac{\partial g_i(x)}{\partial x_j}\right| \leq C_i(\Omega)\, x^{p_{i,j}} + D_i, \quad 0 < x < \infty$$

$$p_{i,j} > 0 \quad = 1,2,\ldots$$

and θ_i is a function with all derivatives continuous and bounded. Let S be a triangulation of Ω such that <u>all vertices of Ω are vertices of the triangulation.</u> We now give the rate of convergence of the p-version of the finite element method for this situation.

<u>Theorem</u>: Suppose u_0, the exact solution to the problem (2.1) (2.2) can be written in the form (2.9), (2.10) (2.11) and let u_p be the finite element approximation. Then, for $k > 1$

$$\|u_0 - u_p\|_{1,\Omega} \leq C(\varepsilon)p^{-\mu+\varepsilon}, \quad \varepsilon > 0, \text{ arbitrary} \quad (2.12)$$

$$\mu = \min(k-1, 2\gamma_i) \quad (2.13)$$

It can also be shown that the estimate (2.12) (2.13) is the best possible. Although the considerations which lead to this estimate are only for the model problem, they apply more generally to the analysis of the behavior of approximations to any function in $H_0^1(\Omega)$ resp. $H^1(\Omega)$.

Thus, in terms of the number of degrees of freedom N, the p-version solution of the singularity problem leads to the estimate

$$\|u_0 - u_p\|_{1,\Omega} \leq C\, N^{-\frac{\mu}{2}+\varepsilon} \quad \varepsilon > 0, \text{ arbitrary}$$

$$\mu = \min(k-1, 2\gamma_i)$$

whereas in the h-version we have the estimate

$$\|u_0 - u_h\|_{1,\Omega} \leq C\, N^{-\frac{\mu}{4}},$$

that is, the uniform p-version has twice the order of convergence of the uniform h-version. Let us remark, however, that when a suitable refinement of the elements is used in the h-version then its convergence rate is, in general, better than in the case of the p-version with fixed mesh. A general theory for combining the theoretical and practical order advantages of both the h- and p-versions is not yet fully developed.

2.2. Illustrations

In order to illustrate the results of the theorems we present two examples. The first is a simple bar problem in one dimension and the numerical results are based on a computer program written specifically for this problem. The second is a problem in two dimensional linear elasticity involving the analysis of a centrally cracked panel. This problem was solved using COMET-X (COnstraint METhod-eXperimental) a general purpose finite element computer program developed at Washington University which implements the p-version of the finite element method.

2.2.1. A One Dimensional (bar) Problem.

Consider the problem: $' = \frac{d}{dx}$, $\Omega = (-1,1)$

$$u'' = -q(x) \quad \text{for } x \in \Omega$$

where the (loading) function $q(x)$ and the (Dirichlet) boundary conditions will be specified later. The energy inner product is

$$B(u,v) = (u,v)_E = \int_{-1}^{1} u'(x)v'(x)\,dx$$

The weak solution $u \epsilon H_0^1(\Omega)$ satisfies

$$(u,v)_E = \int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} q(x)v(x)dx$$

$$\text{for all } v \epsilon H_0^1(\Omega)$$

First, we consider convergence when Ω is not partitioned, i.e. we use only one interval. We choose as basis functions 1, x and

$$\psi_i(x) = \int_{-1}^{x} P_i(t)dt \qquad i \geq 1$$

where $P_i(t)$ is the Legendre polynomial of degree i. If we write

$$u_p(x) = \frac{1-x}{2} u(-1) + \frac{1+x}{2} u(1) + \sum_{i=1}^{p} a_i \psi_i(x)$$

$$p \geq 1$$

it follows from the orthogonality of Legendre polynomials that

$$a_i = \frac{2i+1}{2} \int_{-1}^{1} q(x)\psi_i(x)dx \qquad i = 1,2,\ldots,p$$

Also, denoting the error by $e_p(x) = u(x) - u_p(x)$, it follows that

$$||e_p||_E^2 = ||u - u_p||_E^2 = ||u||_E^2 - ||u_p||_E^2$$

$$= ||\sum_{i=p+1} a_i \psi_i(x)||_E^2$$

$$= \sum_{i=p+1} a_i^2 \frac{2}{2i+1} .$$

If we let $U = \frac{1}{2}||u||_E^2$ denote the strain energy then $U - U_p = ||e_p||_E^2$ is the error in strain energy. We consider two cases:

$$\text{Case A} \quad \frac{du}{dx} = \sqrt{1-x^2},\ q(x) = -\frac{d}{dx}\left(\sqrt{1-x^2}\right),$$

$$u(-1) = -u(1) = \frac{\pi}{4}$$

$$\text{Case B} \quad u(x) = |x|^{3/2}(1-x^2),$$

$$q(x) = -\frac{d^2}{dx^2}\left(|x|^{3/2}(1-x^2)\right),\ u(-1) = u(1) = 0$$

The qualitative difference between the two cases is that in case A the square root singularity in u' is at end points of Ω, whereas in case B it is in its interior. It has been shown[8] that if N denotes the number of degrees of freedom (N=p+1) then as $N \to \infty$

$$\text{in Case A} \quad a_N = O\left(\frac{1}{N^2}\right), \quad ||e_p||_E^2 = O\left(\frac{1}{N^4}\right)$$

$$\text{in Case B} \quad a_N = O\left(\frac{1}{N}\right), \quad ||e_p||_E^2 = O\left(\frac{1}{N^2}\right)$$

This illustrates the importance of locating the singularities at vertices of the finite element mesh in order to obtain the maximal rate of convergence in the p-version, as stated in the second theorem. In order to illustrate this point further and to compare the rates of convergence of the (uniform) h- and p-versions of the finite element method, we plot in Figure 4

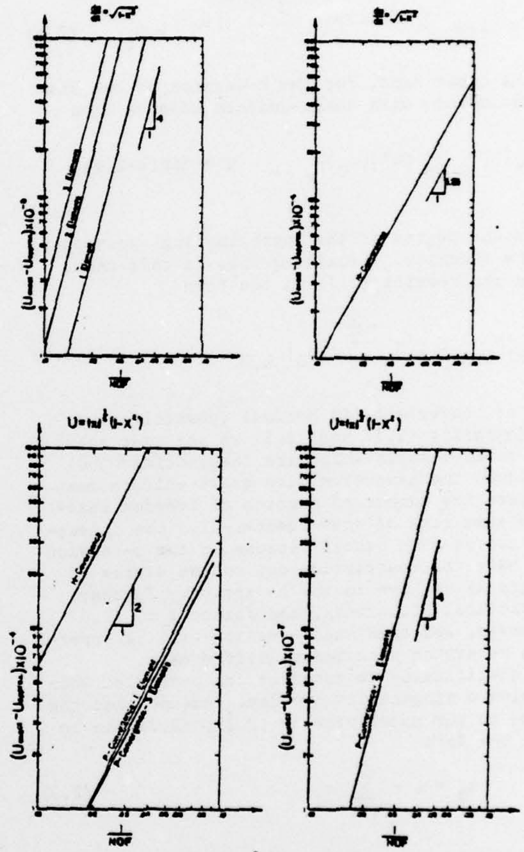


FIGURE 4: CONVERGENCE PLOTS FOR BAR PROBLEM

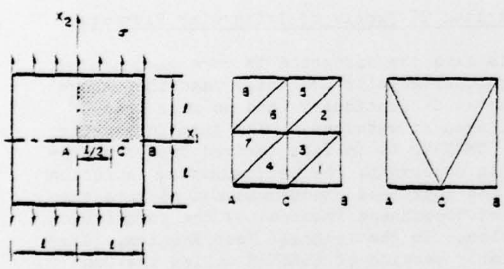
$||e_p||_E^2$ using one, two and three equal intervals for the p-version, and $||e_h||_E^2$ where we have used the notation $e_h(x) = u(x) - u_h(x)$. Linear approximations were used on equal intervals in the h-version. It is known that $||e_h||_E^2 = O((1/N^2)|\ln N|$ for linear elements, and $||e_h||_E^2 = O(1/N^2)$ for quadratic and higher elements. Figure 4 shows in the log scale that the square of the energy error in the case of the p-version has an exponent that is practically 4. In the h-version the asymptotic range has not yet been reached and the rate is about 1.81 instead of 2, where N, the number of degrees of freedom, is denoted by NDF. The results are summarized in Figure 5.

	h-VERSION	p-VERSION					
		ONE INTERVAL		TWO INTERVALS		THREE INTERVALS	
	$\|e_h\|_E^2$	e_N	$\|e_p\|_E^2$	e_N	$\|e_p\|_E^2$	e_N	$\|e_p\|_E^2$
CASE A	$\frac{1}{N^2}(\ln N)$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N^2}$	$\frac{1}{N^3}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$
CASE B	$\frac{1}{N^2}(\ln N)$	$\frac{1}{N}$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\frac{1}{N^3}$	$\frac{1}{N}$	$\frac{1}{N^2}$

FIGURE 5: RATES OF CONVERGENCE FOR THE h-VERSION (LINEAR ELEMENTS) AND THE p-VERSION OF THE FINITE ELEMENT METHOD IN A BAR PROBLEM

In case A the rate of convergence in the p-version remains the same regardless of the number of intervals because the singularity is always at an end-point of an interval. In case B when the singularity is at an end point of an interval (i.e. when there are two intervals) the maximal rate of convergence in the p-version is achieved, whereas when the singularity is in the interior of an interval (i.e. when these are either one or three intervals) the rate of convergence deteriorates.

2.2.1. A Centrally Cracked Panel. Consider the centrally cracked panel shown in Figure 6.



(a) PROBLEM DEFINITION (b) 8 ELEMENT MESH (c) 3 ELEMENT MESH

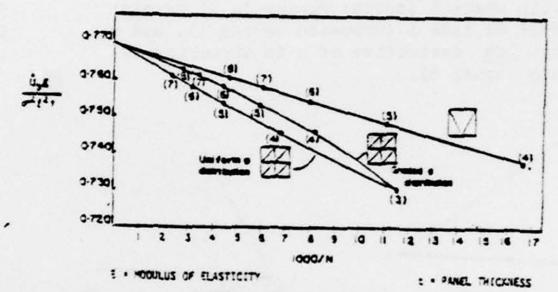
FIGURE 6: CENTRALLY CRACKED PANEL

The displacements have singular behavior in the neighborhood of the crack tip (by symmetry only a quarter of the panel needs to analyzed) in the form $\overline{u} = r^{\gamma}\overline{F}(\theta)$ where r,θ are polar coordinates with respect to the tip, $\gamma = \frac{1}{2}$, and $\overline{F}$ is a smooth function. Two finite element triangulations were used, the eight element mesh and the three element mesh shown in Figure 6. In the eight element mesh the polynomial degrees were distributed in two ways: uniformly and non-uniformly. In the non-uniform or graded distribution the polynomial degrees were greater than p=3 only in crack tip elements (numbered 1,3,4) and the polynomial degree was held constant at p=3 in the remote elements (numbered 5,6,8) and the transition elements (number 2,7). For this problem since $\gamma = \frac{1}{2}$ the theory predicts that the error in strain energy is

$$U - U_p \sim \|u - u_p\|^2_{1,\Omega} \sim C p^{-4\gamma} \sim C p^{-2} \sim C N^{-1}$$

if we neglect the ε in the second theorem and the fact that the cracked panel is not a Lipschitzian domain. The comparable result for the h-version is

$$U - U_h \sim C N^{-\frac{1}{2}}$$



E = MODULUS OF ELASTICITY t = PANEL THICKNESS

FIGURE 7: CENTRALLY CRACKED PANEL
COMPUTED (NORMALIZED) STRAIN ENERGY FOR DIFFERENT p vs. (1000/N)

In Figure 7 the computed strain energy U_p (normalized) is plotted again N^{-1} and the convergence paths are seen to be nearly linear for all $p \geq 4$. Additional details are given for a centrally cracked panel by Szabo[9] and by Szabo and Mehta[10]. In [10] comparative plots of convergence in the p- and h-versions are presented for the case of an edge-cracked panel.

3. Hierarchic Families

In order to implement the p-version of the finite element method, it is necessary to have available a family of finite elements of arbitrary polynomial degree p. Although such families of finite elements have been constructed (by Kratochvil et al,[11] for example), we wish to present a new family which the property that when increasing the degree of the approximating polynomial from p to p+1 as much of the computation as possible is saved from the pth degree approximation. This is clearly a desirable property for efficient computation when using the p-version of the finite element method. More specifically, in this family, basis functions corresponding to an approximation of degree p are a subset of those corresponding to an approximation of degree p+1. Therefore, the stiffness matrix of the element of degree p is a submatrix of the stiffness matrix of the element of degree p+1, and when increasing the degree of approximation from p to p+1 only the added rows and columns of the new stiffness matrix have to be computed. We call a family possesing this property a hierarchic family. COMET-X, the current experimental implementation of the p-version of the finite element method, developed at Washington University, is based on two hierarchic families of conforming triangular finite elements for the analysis of two dimensional problems in linear elasticity. One family enforces C^0 continuity across interelement boundaries for problems which require solutions in $H_0^1(\Omega)$ (planar elasticity), and the other

enforces C^1 continuity across interelement boundaries for problems which require solutions in $H_0^2(\Omega)$ (plate bending).

3.1. Hierarchic C^0 Family of Triangular Elements

This family consists of complete polynomials of degree $p \geq 2$ defined either in terms of nodal variables or (nodeless) basis functions. The nodal variables are divided into two classes: external nodal variables used to enforce global C^0 continuity and internal nodal variables which are added to complete the polynomial of degree p.

Let u denote the approximation to the displacement field, and let u_{s_i} denote the deviative of u in direction of side i. The external nodal variables for each $p \geq 2$ are: $u(i)$, $u_{s_i^j}(i')$ i, i' = 1,2,3, j = 2,3,...p where i denotes vertex i, i' denotes the midpoint of side i (opposite vertex i), and $u_{s_i^j}$ denotes the j*th* derivative of u in direction of side i (see Figure 8).

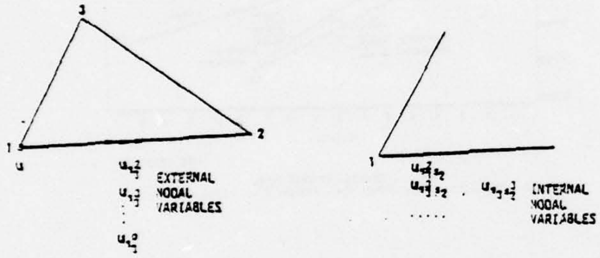


FIGURE 8: HIERARCHIC C^0 TRIANGULAR ELEMENT, $P \geq 2$

For each $p \geq 2$, $\frac{1}{2}(p-1)(p-2)$ internal nodal variables defined as derivatives of order j, j = 3,...,p evaluated at one vertex, are added to give a complete polynormal of degree p[12]. The basis functions for these external nodal variables expressed in natural (triangular) coordinates L_1, L_2, L_3 (see Peano[13], for example, for the definition) are, for vertex 1 and side 1

$$N_{u(1)} = L_1$$

$$N_{u_{s_1^j}}(1') = \begin{cases} \frac{1}{j!}\left(\frac{\ell_1}{2}\right)^j \left[(L_2 - L_3)^j - (L_2 + L_3)^j\right] & \text{if } j \text{ is even} \\ \frac{\ell_1}{2j}\,(L_2 - L_3)N_1^{(j-1)} & \text{if } j \text{ is odd} \end{cases}$$

where ℓ_1 is the length of side 1. The expressions for the other vertices and sides are obtained by cyclic index permutation.

In order to illustrate the difference between non-hierarchic and hierarchic finite elements, let us compare the two when p=2. For non-hierarchic finite elements the nodal variables for the linear approximations are u(i), i = 1,2,3, and for the quadratic approximation the nodal variables u(i') i = 1,2,3 are added. The basis functions for these nodal variables are then: in the linear approximation L_i, i = 1,2,3 corresponds to u(i); in the quadratic approximation $L_i(2L_i - 1)$ is the basis function for u(i) i = 1,2,3 and $4L_iL_{i+1}$ is the basis function for u(i') i = 1,2,3. Thus, the basis function for u(i) changes when increasing the degree of the approximating polynomial from linear to quadratic. For hierarchic elements, however, in the linear approximation L_i is the basis function for u(i) i = 1,2,3 in __both__ the linear and quadratic approximations and $(1/2)L_iL_{i+1}$ is the basis function for $u_{s_i^2}(i')$ in the quadratic approximation. Therefore, the basic functions for the linear approximation are unchanged when increasing the degree of approximation to quadratic. It follows then that the stiffness matrix corresponding to the linear approximation is a submatrix of the stiffness matrix corresponding to the quadratic approximation for the hierarchic family.

It is also possible to construct hierarchic basis functions which do not correspond to nodal variables. In this case the external nodes used to enforce global C^0 continuity are:

L_1, L_2, L_3 corresponding to vertices 1,2,3 respectively

$L_1^jL_2 - L_1(-L_2)^j$, $L_2^jL_3 - L_2(-L_3)^j$, $L_3^jL_1 - L_3(-L_1)^j$,

corresponding to sides 1,2,3, respectively

$j = 1,2,\ldots,p$

The internal basis functions are (j-2) independent polynomials each of which contains factor $L_1L_2L_3$ (so that they vanish on the boundary of the triangle), for j = 3,...,p (see Peano[13,14]).

3.2. Hierarchic C^1 Family of Triangular Elements

In this case the situation is more complicated. The hierarchic triangular C^0 family described above enforces global C^0 continuity (and no more than C^0 continuity, even at vertices). The term Constraint Method, in COMET-X, is in fact derived from the property that it constrains the approximating solution to satisfy the degree of smoothness and no more than the degree of smoothness required by the formulation of the problem. In the Lockheed Test Problem, (Figure 1) an early version of COMET-X called the Constraint Method was used with the excellent results shown in Figures 2 and 3. It is the property that the constraint method does not enforce more than C^1 continuity even at the re-entrant corner that contributed to the good results. Although it is possible to construct a hierarchic C^1 triangular family, it was proved by Peano[13, 14], that in order to enforce global C^1 continuity and no more than global C^1 continuity even at vertices, certain additional constraint equations must be satisfied. Consider, for example, a vertex of a triangular element e and let s_1 and s_2 be coordinates along the two sides which meet at the vertex (see Figure 9).

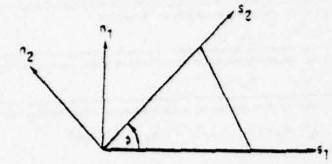


FIGURE 9: DIAGRAM FOR CONSTRAINT EQUATION

Let $(\partial/\partial n_i)$ $i = 1,2$, be the derivative in the direction normal to side i. Then in order for the approximation w to be in C^1 at the vertex the following constraint must hold:

$$\frac{\partial^2 w}{\partial s_1^2}\cos\phi + \frac{\partial^2 w}{\partial s_1 \partial n_1}\sin\phi$$

$$= \frac{\partial^2 w}{\partial s_2^2}\cos\phi - \frac{\partial^2 w}{\partial s_2 \partial n_2}\sin\phi. \qquad (3.1)$$

Constraints of the form (3.1) must be satisfied at all vertices of the triangulation.

The hierarchic C^1 family of triangular elements of order $p \geq 5$, uses for external nodal variables values of w, its first derivatives and its second tangential and mixed (normal-tangential) derivatives at vertices, and derivatives or order ≥ 5 at midside nodes[12] (See Figure 10) for the quintic hierarchic C^1 element). For each $6 \leq j \leq p$, the jth order tangential derivatives at midsides, and a mixed jth order derivative (j-1 tangential derivatives, one normal derivative) at midsides are used to enforce C^1 continuity.

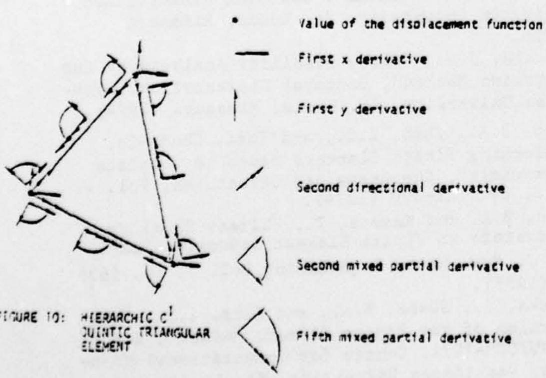


FIGURE 10: HIERARCHIC C^1 QUINTIC TRIANGULAR ELEMENT

Basis functions for these external nodal values as well as for internal nodes all of which contain a factor $(L_1L_2L_3)^2$ are given in [13, 14]. Several procedures are available to enforce the constraint equations (3.1):

(1) A specially devised global assembly process which reduces the assembly of elements to a standard finite element assembly procedure [14,15].
(2) Creation of super elements (or macro elements) of arbitrary degree $p \geq 5$. In these macro elements constraints are satisfied within the macro element leaving external nodal variables on the boundary to be freely assembled [14, 15].
(3) Adding newly constructed corrective rational functions to the basis [13, 14]. These rational functions modify the smoothness of the approximations at vertices (while preserving C^1 continuity) but permit a free assembly without enforcing constraints. A method has been devised for integrating these rational functions over triangles directly without recourse to numerical quadrative [16].

3.3. Hierarchic C^0 Family of Rectangular Elements

Again we choose as nodal variables the values of the approximation u at vertices and higher tangential derivatives of degree j, $2 \leq j \leq p$ at midside nodes. First observe that the polynomial

$$Q_j(\xi) = \begin{cases} \frac{1}{j!}(\xi^2 - 1) & j \geq 2, \text{ even} \\ \frac{\xi}{j} Q_{j-1}(\xi) & j \geq 3, \text{ odd} \end{cases}$$

satisfies

$$Q_j(\pm 1) = 0 \qquad Q_j^{(i)}(0) = 0 \qquad i = 2,\ldots,j-1$$

$$Q_j^{(j)}(0) = 1.$$

Now, consider the square of side 2 shown in Figure 11.

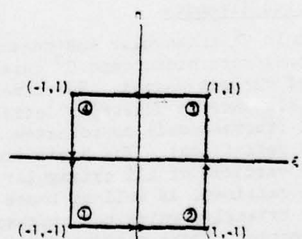


FIGURE 11: HIERARCHIC QUADRATIC C^0 SQUARE ELEMENT

Basis functions $N_{u(i)}$ corresponding to the nodal variables u(i) $i = 1,2,3,4$ are

$$N_{u(1)} = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_{u(3)} = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_{u(2)} = \frac{1}{4}(1+\xi)(1-\eta) \qquad N_{u(4)} = \frac{1}{4}(1-\xi)(1+\eta) \qquad (3.2)$$

and it is easily seen that these basis functions span

the same space as $1,\xi,\eta,\xi\eta$ i.e. they contain the complete linear polynomial. Also these nodal variables enforce C^0 continuity across sides. Now denoting by (ij) the midpoint of side ij, basis functions corresponding to the nodal variables $u_{\xi\xi}(12)$, $u_{\eta\eta}(23)$, $u_{\xi\xi}(34)$, $u_{\eta\eta}(41)$ are

$$N_{u_{\xi\xi}}(12) = Q_2(\xi)(1-\eta) \qquad N_{u_{\xi\xi}}(34) = Q_2(-\xi)(1+\eta)$$
$$N_{u_{\eta\eta}}(23) = (1+\xi)Q_2(\eta) \qquad N_{u_{\eta\eta}}(41) = (1-\xi)Q_2(-\eta) \tag{3.3}$$

and these basis functions added to the ones in (3.2) span the same space as $1,\xi,\eta,\xi^2,\xi\eta,\eta^2,\xi^2\eta,\xi\eta^2$, i.e. they contain a complete quadratic and they enforce C^0 continuity along sides. The basis functions (3.2) are taken for the hierarchic rectangular C^0 linear element, and those in (3.2) and (3.3) for the quadratic element.

For $j \geq 3$, we have

$$N_{u_{\xi j}}(12) = Q_j(\xi)(1-\eta) \qquad N_{u_{\xi j}}(34) = Q_j(-\xi)(1+\eta)$$

$$N_{u_{\eta j}}(23) = (1+\xi)Q_j(\eta) \qquad N_{u_{\eta j}}(41) = (1-\xi)Q_j(-\eta)$$

as the basis function for jth order tangential derivatives at midsides, and for $j \geq 4$ we add the internal modes

$$(1-\xi^2)(1-\eta^2)\xi^{j-4-i}\eta^i \quad i = 0,\ldots,j-4.$$

These basis functions span the same space as a complete polynomial of degree and the two monomials of degree $j+1$, $\xi^j\eta,\xi\eta^j$.

Thus, the hierarchic C^0 square element of degree $p \geq 2$ has $(1/2)(p+1)(p+2)+2$ basis functions, two more than the dimension of the complete polynomial of degree p. The two extra terms correspond to $\xi^p\eta,\xi\eta^p$. By scaling the sides of the square the elements are easily transformed into rectangular ones.

3.4. Hierarchic C^0 Solid Elements

Using the hierarchic C^0 triangular and rectangular elements, we can construct hierarchic C^0 three dimensional elements of various shapes. For exmaple, using only triangular elements we construct tetrahedral C^0 elements in natural (tetrahedral) coordinates (see 17, for example, for a definition). The basis functions corresponding to vertices of all triangular faces of the tetrahedron are retained, as well as those corresponding to sides of triangles which now correspond to edges of the tetrahedron. Those corresponding to internal nodes of triangles now correspond to face modes, and internal modes of the tetrahedron all contain the factor $L_1L_2L_3L_4$ i.e. they vanish on all faces of the tetrahedron (see Figure 12, for the first four hierarchic tetrahedral C^0 elements).

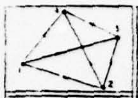


FIGURE 12

NODAL VARIABLES AND BASIS FUNCTIONS FOR THE FIRST FOUR HIERARCHIC TETRAHEDRAL C^0 ELEMENTS

1. LINEAR	4 TERMS: VALUES OF THE APPROXIMATING FUNCTION AT THE VERTICES. $N_1 = L_1,\ N_2 = L_2,\ N_3 = L_3,\ N_4 = L_4$
2. QUADRATIC	6 ADDITIONAL TERMS: SECOND DERIVATIVES AT THE MIDPOINTS OF EDGES. NORMALIZING FACTOR: -1/2 $N_5 = L_1L_2,\ N_6 = L_2L_3,\ \ldots\ N_{10} = L_1L_4$
3. CUBIC	10 ADDITIONAL TERMS: SIX THIRD DERIVATIVES AT THE MIDPOINTS OF EDGES. NORMALIZING FACTOR: 1/12. $N_{11} = L_1^2L_2 - L_1L_2^2,\ N_{12} = L_2^2L_3 - L_2L_3^2,\ \ldots\ N_{16} = L_1^2L_4 - L_1L_4^2$ FOUR FACE MODES: $N_{17} = L_1L_2L_3,\ N_{18} = L_1L_2L_4,\ N_{19} = L_1L_3L_4,\ N_{20} = L_2L_3L_4$
4. QUARTIC	15 ADDITIONAL TERMS: SIX FOURTH DERIVATIVES AT THE MIDPOINTS OF EDGES. NORMALIZING FACTOR: - 1/48: $N_{21} = L_1^3L_2 + L_1L_2^3,\ N_{22} = L_2^3L_3 + L_2L_3^3,\ \ldots\ N_{26} = L_1^3L_4 + L_1L_4^3$ EIGHT FACE MODES: $N_{27} = L_1^2L_2L_3,\ N_{28} = L_1L_2^2L_3,\ N_{29} = L_1^2L_2L_4,\ N_{30} = L_1L_2^2L_4,\ N_{31} = L_1^2L_3L_4$ $N_{32} = L_1L_3^2L_4,\ N_{33} = L_2^2L_3L_4,\ N_{34} = L_2L_3^2L_4$ ONE INTERNAL MODE: $N_{35} = L_1L_2L_3L_4$

Using the hierarchic rectangular C^0 elements, it is possible to construct hierarchic C^0 brick elements, and using both the hierarchic C^0 rectangular and C^0 triangular elements it is possible to construct hierarchic C^0 triangular prismatic elements.[18]

4. Acknowledgements

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5. References

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Comparative Rates of h- and p-convergence in the Finite Element Analysis of a Model Bar Problem

The conventional approach to finite element stress analysis of a body defined by a polygonal domain Ω (in two dimensions) is to triangulate Ω and to seek accuracy by letting h, the maximum diameter of all elements in the triangulation, tend to zero. This approach, called h-convergence, has been the subject of intensive investigation. Another approach which is being developed at Washington University is to fix the triangulation of Ω and to let p, the degree of the complete, conforming, approximating polynomial over each triangle, tend to infinity. Extensive numerical tests have shown that the second approach, called p-convergence, is considerably more accurate than the first, even in problems whose solutions have singularities such as cracks or corners. In order to illustrate the comparative rates of convergence, a model (one-dimensional) bar problem is studied. Asymptotic analysis leads to expressions for the rates of convergence in the two approaches, when the solution possesses a singularity which is known a priori. It is demonstrated that the order of p-convergence is twice that of h-convergence, provided that the singularity is located at some node of a finite element.

x

x